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THE CLASS OF ORDER L-WEAKLY AND ORDER M-WEAKLY DEMICOMPACT OPERATORS

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Abstract
This work introduces two new classes of operators, called order L-weakly demicompact (order LWDC) and order M-weakly demicompact (order MWDC). We give several results that compare order LW-compact and order MW-compact operators with the corresponding demicompact operators. We also define order power MW-compact operators and prove that they belong to the class of order MWDC operators. In addition, we show that this class is preserved under perturbations by order M-weakly compact operators.

Keywords Order M-weakly demicompact · Banach lattice · Order L-weakly demicompact · Order M-weakly compact · Order L-weakly compact

Mathematics Subject Classification (2010) 47B60 · 47B65 · 46B42 · 46B40 · 46A40

Introduction

The study of compact and demicompact operators occupies an important position in functional analysis and operator theory. The notion of demicompactness was first introduced by Petryshyn [17] in the context of Hilbert spaces to generalize the classical concept of compactness while preserving enough compact-type properties to guarantee the existence of fixed points. Building on Petryshyn's work, the theory of demicompact operators has been substantially developed, particularly in [3] showing that demicompact operators on Banach spaces satisfy several key perturbation theorems. This concept has been widely used and applied to the study of spectral stability [6, 8, 13] and perturbation theory [5]. Also, the authors in [9, 10] investigate a new class via the demicompactness theory. Recently, the classical results of Petryshyn in Fredholm theory have been extended to Banach spaces with order structures [12], lattice-normed spaces [11], and Banach lattices [2, 7].

The introduction of order and lattice structures in the framework of Banach lattices has led to new perspectives on weak compactness. In [15], the author studied the notions of L-weakly compact (for short LW) and M-weakly (for short MW) compact subsets and operators. These classes of operators generalize the usual compact operators by incorporating the lattice order, and they have been extensively used to characterize weak compactness in ordered Banach spaces.

The order structure of Banach lattices also motivated the development of new compactness notions. The class of order weakly compact operators was first studied by Dodds [4]. This class requires that, for each $u \in \mathbb{E}^+$, the image of the order interval $[0, u]$ under an operator is relatively weakly compact. More recently, Lhaimer et al. [14] refined this idea by introducing order L-weakly compact (for short order LW-compact) and order M-weakly compact (for short order MW-compact) operators.

In [14], the authors characterize and establish properties of this class within Banach lattices. They focus on compact-type behavior under order constraints. This research makes the following key contributions:

(i) Study how order LW-compact and order MW-compact operators are linked to their demicompact versions.
(ii) Several examples and counterexamples are presented in order to illustrate that the classes of order LWDC and order MWDC properly contain the corresponding classes of order LW-compact and order MW-compact operators.

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(iii) Define and characterize order power MW-compact operators and prove that they are order MWDC.
(vi) Demonstrate the stability of the introduced classes under order MW-compact perturbations.
(v) Analyze the behavior of order LWDC matrix operators acting on direct sums of Banach lattices.

We now describe the plan of the paper. The background material and required results are collected in the "Preliminaries" section. The notions of order LWDC and order MWDC are introduced in the "Main results" section. Further, we establish structural and stability results connecting these new classes with their compact counterparts. Finally, we analyze the behavior of order LWDC operator matrices.

Preliminaries

First, recall some basic facts. We let X and Y be Banach spaces. We let $\mathbb{E}$ and $\mathbb{F}$ be Banach lattices. The closed unit ball of X is written $\mathbb{U}_X$. An ordered vector space is called a vector lattice if every pair x, y has a supremum $\sup(x, y)$. A space $(\mathbb{E}, \|\cdot\|)$ is a Banach lattice when it is a Banach space, a vector lattice, and $|x| \leq |y|$ gives $\|x\| \leq \|y\|$.

The dual $\mathbb{E}^*$ of a Banach lattice $\mathbb{E}$, with dual norm and dual order, is also a Banach lattice. The positive cone, namely $\mathbb{E}^+$, is the set of all non-negative elements. We use the word operator for any bounded linear map between Banach spaces. We denote $\mathcal{B}(\mathbb{E}, \mathbb{F})$ the space of all bounded operators from $\mathbb{E}$ to $\mathbb{F}$.

The symbol $x_\beta \downarrow 0$ is used to say that (x_β) is a decreasing family in $\mathbb{E}$, that a greatest lower bound exists, and that this bound is zero. For a Banach lattice $\mathbb{E}$, the norm of $\mathbb{E}$ is called order continuous when every family (x_β) satisfies that $\|x_\beta\| \to 0$. We now move on to define LW-compactness for sets and the related operator classes introduced in [15].

Definition 2.1 [15] A bounded set D in a Banach lattice is LW-compact whenever any disjoint sequence contained in its solid part converges to zero in norm.

Definition 2.2 (1) $A \in \mathcal{B}(\mathbb{E}, \mathbb{F})$ is LW-compact when $A(U_{\mathbb{E}})$ is an LW-compact set in $\mathbb{F}$.
(2) An operator $A : \mathbb{E} \to Y$ is MW-compact if the sequence (Au_n) converges to zero in norm, for each disjoint bounded sequence (u_n).

Sequentially, Lhaimer et al. extended Definition 2.2 to an ordered framework.

Definition 2.3 [14] $A : \mathbb{E} \to \mathbb{F}$ is order LW-compact if the image of the order interval $[0, y]$ under A is an LW-compact set, for each positive element y.

Definition 2.4 [14] A bounded operator $A : \mathbb{E} \to \mathbb{F}$ is order MW-compact when the following is true: the real numbers $g_n(A(u_n))$ go to zero any order bounded sequence (g_n) in $\mathbb{F}^*$ and for any disjoint sequence (u_n) in $U_{\mathbb{E}}$.

Definition 2.5 [17] Let $B \in \mathcal{L}(X)$. Whenever a bounded sequence (u_n) satisfies $u_n - Bu_n \to 0$ in norm, the sequence (z_n) has a norm convergent subsequence. In this case, we call B demicompact.

Main results

We define a new class of operators as follows.

Definition 3.1 Let $A : \mathbb{E} \to \mathbb{E}$ be an operator. We say that A is order L-weakly demicompact (for short order LWDC) when, for any order bounded sequence (u_n) in $U_{\mathbb{E}}$, the LW-compactness of the set $\{u_n - Au_n : n \in \mathbb{N}\}$ implies the LW-compactness of $\{u_n : n \in \mathbb{N}\}$.

We now establish a fundamental relationship between order LW-compact and the newly introduced class of order LWDC operators.

Proposition 3.1 *Every order LW-compact operator $A : \mathbb{E} \to \mathbb{E}$ is order LWDC.*

Proof Let (f_k) be a disjoint sequence in the solid hull of $\{u_n : n \in \mathbb{N}\}$. To prove LW-compactness, it is enough to show that $\|f_k\| \to 0$. Hence, we take a subsequence (u_{n_k}) of (u_n) such that $|f_k| \leq |u_{n_k}|$ for every $k \in \mathbb{N}$, where (u_n) is an order bounded in $\mathbb{E}^+$ with $\{u_n - Au_n : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}$. Since

$$|f_k| \leq |u_{n_k} - Au_{n_k}| + |Au_{n_k}|, \ \ k \in \mathbb{N},$$

the Riesz decomposition property (see [1]) guarantees the existence of elements $f_{k1}, f_{k2} \in E$ such that

$$f_k = f_{k1} + f_{k2}, \quad |f_{k1}| \leq |u_{n_k} - Au_{n_k}|, \quad |f_{k2}| \leq |Au_{n_k}|.$$

From the first inequality, and the fact that the disjoint sequence (f_{k1}) belongs to the solid hull of $\{u_n - Au_n : n \in \mathbb{N}\}$, which is LW-compact, it follows that $\|f_{k1}\| \to 0$. Similarly, the second inequality shows that (f_{k2}) is disjoint in the solid hull of $\{Au_n : n \in \mathbb{N}\}$. Because A is order LW-compact, we also have $\|f_{k2}\| \to 0$. Hence,

$$\|f_k\| \leq \|f_{k1}\| + \|f_{k2}\| \to 0,$$

which proves that $\{u_n : n \in \mathbb{N}\}$ is LW-compact. Therefore, A is order LWDC. □

Remark 3.1 Every LW-compact is an order LWDC operator. But the opposite is not always true. As an example, take the identity operator $I_{\ell_1} : \ell_1 \to \ell_1$. One can see that for $\beta \neq 1$, the operator βI_{ℓ_1} is order LWDC. However, U_{ℓ_1} is not relatively weakly compact. So I_{ℓ_1} is not LW-compact (see [16]). Consequently, βI_{ℓ_1} is not LW-compact.

Example 3.1 Let $1 \leq p < \infty$ and consider the Banach lattice $\mathbb{E} = L^p([0, 1])$ equipped with the Lebesgue measure and the pointwise almost everywhere order. Let $m \in L^\infty([0, 1])$ be a bounded measurable function with $\|m\|_\infty \leq 1$. Define the multiplication operator $M_m : \mathbb{E} \to \mathbb{E}$ by

$$(M_m f)(x) = m(x) f(x), \quad x \in [0, 1], \quad f \in L^p([0, 1]).$$

This is a bounded linear operator with $\|M_m\| \leq \|m\|_\infty \leq 1$. We claim that M_m is order LWDC. To see this, let (f_n) be an order bounded sequence in the positive part of the unit ball of $L^p([0, 1])$, i.e., $0 \leq f_n \leq u$ for some $u \in L^p([0, 1])^+$ and all n. Assume that the set $\{f_n - M_m f_n : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}$. Note that $f_n - M_m f_n = (1 - m) f_n$. Since $\{(1 - m) f_n : n \in \mathbb{N}\}$ is LW-compact and M_m is a positive operator (as $|m| \leq 1$), one can use the lattice properties and order continuity of the norm in L^p to deduce that the sequence (f_n) itself satisfies the LW-compactness condition. More precisely, disjoint sequences in the solid hull of $\{(1 - m) f_n\}$ converge to zero in norm. Because multiplication operators preserve disjointness in an appropriate sense (up to the support of $1 - m$), and using the fact that L^p spaces have order continuous norms, it follows that disjoint sequences extracted from the solid hull of (f_n) also converge to zero in norm. Thus, M_m is order LWDC.

However, M_m is not necessarily order LW-compact. For instance, take $m(x) \equiv 1$ on a set of positive measure (or $m \equiv \beta$ with $|\beta| < 1$ but close to 1). Then, the image of the order interval $[0, \chi_{[0,1]}]$ under M_m contains functions that fail to have relatively weakly compact images in the usual sense (e.g., characteristic functions of shrinking intervals where $m \approx 1$). This shows that M_m belongs to the new class but lies strictly outside the order LW-compact operators.

Remark 3.2 Similar constructions work for weighted multiplication operators on sequence spaces ℓ_p (with a bounded weight sequence (w_n) satisfying $|w_n| \leq 1$). These examples illustrate the strict inclusion of order LWDC classes over their compact counterparts and provide natural non-trivial illustrations in classical function spaces.

An operator $A : \mathbb{E} \to \mathbb{E}$ is order power LW-compact if there is a positive integer m such that the power A^m is order LW-compact.

We first present the following lemma which gives a condition that we will use later. It allows us to study order power LW-compact operators.

Lemma 3.1 [18] *Consider a regular operator A from $\mathbb{E}$ into $\mathbb{F}$ which has an order continuous norm. For every LW-compact set M, the set $A(M)$ is LW-compact in $\mathbb{F}$.*

Proposition 3.2 *Every regular operator A from $\mathbb{E}$ with order continuous norm, that is, order power LW-compact, is order LWDC.*

Proof Let us fix an operator $A : \mathbb{E} \to \mathbb{E}$. Take a bounded order sequence (u_n) in $\mathbb{E}^+$ with $\{u_n - Au_n : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}$. From the assumption on A, one can find an integer $q \geq 1$ with the property that A^q is order LW-compact. For this reason, the set $\{A^q u_n : n \in \mathbb{N}\}$ is LW-compact. At the same time, the operator A^{q-1} is regular. Hence, by Lemma 3.1, the set $\{A^{q-1} u_n - A^q u_n : n \in \mathbb{N}\}$ is also LW-compact. Moreover, we have

$$\{A^{q-1} u_n : n \in \mathbb{N}\} \subset \{A^{q-1} u_n - A^q u_n : n \in \mathbb{N}\} + \{A^q u_n : n \in \mathbb{N}\}.$$

As a result, the set $\{A^{q-1} u_n : n \in \mathbb{N}\}$ is LW-compact. We apply the same argument step by step for the powers $q-2, q-3, \ldots, 1$. Since each operator A^i is regular, Lemma 3.1 gives that the sets $\{A^i u_n - A^{i+1} u_n : n \in \mathbb{N}\}$ are LW-compact. From the inclusion

$$\{A^i u_n : n \in \mathbb{N}\} \subset \{A^i u_n - A^{i+1} u_n : n \in \mathbb{N}\} + \{A^{i+1} u_n : n \in \mathbb{N}\},$$

we conclude by induction that $\{A^i u_n : n \in \mathbb{N}\}$ is LW-compact. Thus, $\{Au_n : n \in \mathbb{N}\}$ is LW-compact. This shows that A is order LW-compact. By Proposition 3.1, the operator A is order LWDC. □

Example 3.2 Let $1 \leq p < \infty$ and let $P : L^p([0,1]) \longrightarrow L^p([0,1])$ be defined by

$$(Pf)(t) = \chi_{[0,\frac{1}{2}]}(t) f(t), \;\; f \in L^p([0,1]).$$

Then, P is a bounded positive projection and $P^2 = P$. Since the range $R(P) = L^p\left([0,\frac{1}{2}]\right)$ is infinite dimensional, P is not compact, and consequently, it is not order LW-compact.
On the other hand, $L^p([0,1])$ has order continuous norm; therefore, by Proposition 3.2, P is order LWDC.

We focus on matrix operators on direct sums of Banach lattices. Define $\mathcal{A} : \mathbb{E} \to \mathbb{E}$ by

$$\mathcal{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{pmatrix}, \tag{3.1}$$

where $\mathbb{E} = \bigoplus_{i=1}^m \mathbb{E}_i$. All entries $A_{i,j} : \mathbb{E}_j \to \mathbb{E}_i$ are bounded linear operators.

Proposition 3.3 *Assume that each diagonal operator $A_{i,i} : \mathbb{E}_i \to \mathbb{E}_i$ is order LWDC. Then, the diagonal operator, defined by*

$$\mathcal{D} = \begin{pmatrix} A_{1,1} & 0 & \cdots & 0 \\ 0 & A_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{m,m} \end{pmatrix},$$

is order LWDC.

Proof Take a sequence $Y_n = (y_n^1, y_n^2, \ldots, y_n^m), \quad n \in \mathbb{N}$, which is order bounded in $\mathbb{E}^+$. Assume that $\{Y_n - \mathcal{D}Y_n : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}$. Our goal is to show that the set $\{Y_n : n \in \mathbb{N}\}$ is also LW-compact in $\mathbb{E}$. By the definition of $\mathcal{D}$, we have

$$Y_n - \mathcal{D}Y_n = \begin{pmatrix} y_n^1 - A_{1,1} y_n^1 \\ y_n^2 - A_{2,2} y_n^2 \\ \vdots \\ y_n^m - A_{m,m} y_n^m \end{pmatrix}, \qquad n \in \mathbb{N}.$$

This shows that, for each $i = 1, 2, \ldots, m$, the set $\{y_n^i - A_{i,i} y_n^i : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_i$. Because each diagonal operator $A_{i,i}$ is order LWDC, we obtain that $\{y_n^i : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_i$ for every $i = 1, \ldots, m$. As a result, the product set $\{Y_n = (y_n^1, y_n^2, \ldots, y_n^m) : \; n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}$. This proves that the diagonal operator $\mathcal{D}$ is order LWDC. □

Proposition 3.4 *Consider the space $\mathbb{E} = \bigoplus_{i=1}^m \mathbb{E}_i$, where each $\mathbb{E}_i$ is a Banach lattice. Let $A_{i,j} : \mathbb{E}_j \to \mathbb{E}_i$ be operators for all i, j. Assume that the conditions are satisfied:*

(1) The norm of $\mathbb{E}$ is order continuous.
(2) Each $A_{i,i}$ is order LWDC.
(3) $A_{i,j}$ is LW-compact when $i < j$.
(4) $A_{i,j}$ is regular when $i > j$.

The matrix operator $\mathcal{A}$ in representation (3.1) is order LWDC.

Proof The norm of each space $\mathbb{E}_i$ is order continuous for $i = 1, \ldots, m$, because the order continuity of the norm of $\mathbb{E}$. Let $Y_n = (y_n^1, y_n^2, \ldots, y_n^m), \quad n \in \mathbb{N}$, be an order bounded sequence in $\mathbb{E}^+$. Assume that $\{Y_n - \mathcal{A}Y_n : n \in \mathbb{N}\}$ is LW-compact. We want to prove that $\{Y_n : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}$. It is enough to show that, for every index i, the set $\{y_n^i : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_i$. For each $n \in \mathbb{N}$, we can write

$$Y_n - \mathcal{A}Y_n = \begin{pmatrix} y_n^1 - A_{1,1} y_n^1 - A_{1,2} y_n^2 - \cdots - A_{1,m} y_n^m \\ y_n^2 - A_{2,1} y_n^1 - A_{2,2} y_n^2 - \cdots - A_{2,m} y_n^m \\ \vdots \\ y_n^m - A_{m,1} y_n^1 - A_{m,2} y_n^2 - \cdots - A_{m,m} y_n^m \end{pmatrix}.$$

Hence, for each $i = 1, \ldots, m$, the set $\{y_n^i - A_{i,1} y_n^1 - \cdots - A_{i,m} y_n^m : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_i$.

Case $i = 1$. The operators $A_{1,j}$ are LW-compact for all $j = 2, \ldots, m$. For this reason, the set $\{A_{1,2}y_n^2 + \cdots + A_{1,m}y_n^m : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_1$. Also, we have

$$\{y_n^1 - A_{1,1}y_n^1 : n \in \mathbb{N}\} \subset \{y_n^1 - A_{1,1}y_n^1 - \cdots - A_{1,m}y_n^m : n \in \mathbb{N}\} + \{A_{1,2}y_n^2 + \cdots + A_{1,m}y_n^m : n \in \mathbb{N}\}.$$

Therefore, the set $\{y_n^1 - A_{1,1}y_n^1 : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_1$. Since $A_{1,1}$ is order LWDC, it yields that $\{y_n^1 : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_1$.

Case $2 \le i \le m-1$. For every $j < i$, the operator $A_{i,j}$ is regular. By Lemma 3.1, the sets $\{A_{i,j}y_n^j : n \in \mathbb{N}\}$ are LW-compact in $\mathbb{E}_i$. In addition, for $j > i$, the operators $A_{i,j}$ are LW-compact. So the set $\{A_{i,i+1}y_n^{i+1} + \cdots + A_{i,m}y_n^m : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_i$.

We also have

$$\{y_n^i - A_{i,i}y_n^i : n \in \mathbb{N}\} \subset \{y_n^i - A_{i,1}y_n^1 - \cdots - A_{i,m}y_n^m : n \in \mathbb{N}\} + \sum_{j=1}^{i-1}\{A_{i,j}y_n^j : n \in \mathbb{N}\}$$
$$+ \{A_{i,i+1}y_n^{i+1} + \cdots + A_{i,m}y_n^m : n \in \mathbb{N}\}.$$

Hence, the set $\{y_n^i - A_{i,i}y_n^i : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_i$. Because $A_{i,i}$ is order LWDC, we obtain that $\{y_n^i : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_i$.

Case $i = m$. For all $j = 1, \ldots, m-1$, the operator $A_{m,j}$ is regular. Thus, the sets $\{A_{m,j}y_n^j : n \in \mathbb{N}\}$ are LW-compact in $\mathbb{E}_m$. We also have the inclusion

$$\{y_n^m - A_{m,m}y_n^m : n \in \mathbb{N}\} \subset \sum_{j=1}^{m-1}\{A_{m,j}y_n^j : n \in \mathbb{N}\} + \{y_n^m - A_{m,1}y_n^1 - \cdots - A_{m,m}y_n^m : n \in \mathbb{N}\}.$$

Therefore, the set $\{y_n^m - A_{m,m}y_n^m : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_m$. Since $A_{m,m}$ is order LWDC, we conclude that $\{y_n^m : n \in \mathbb{N}\}$ is LW-compact in $\mathbb{E}_m$.

Putting all cases together, each coordinate set $\{y_n^i : n \in \mathbb{N}\}$ is LW-compact. Hence, the set $\{Y_n : n \in \mathbb{N}\}$ is LW-compact. □

Example 3.3 Consider the right shift operator $S : \ell_1 \longrightarrow \ell_1$ defined by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. For the canonical basis (e_n) of ℓ_1, we have $S(e_n) = e_{n+1},\ n \ge 1$. The sequence (e_{n+1}) is disjoint and satisfies $\|e_{n+1}\|_1 = 1,\ n \ge 1$. Hence, S is not LW-compact. Moreover, $(I - S)e_n = e_n - e_{n+1}$, and therefore, $\|(I - S)e_n\|_1 = 2,\ n \ge 1$. Thus, $(I - S)e_n$ does not converge to zero in norm, showing that S is not demicompact. In particular, S does not belong to the class of order LWDC.

This example shows that the class of order LWDC operators is a proper subclass of the class of all bounded operators.

The following lemma will be instrumental in establishing the next result, which provides a sequence description of order LWDC operators.

Lemma 3.2 [1, Theorem 5.63] *Let $M \subset \mathbb{E}$ and $N \subset \mathbb{E}^*$ be bounded and nonempty. Disjoint sequences from the solid hull of M converge uniformly to zero on N if and only if disjoint sequences from the solid hull of N converge uniformly to zero on M.*

Theorem 3.1 *Consider an operator $A : \mathbb{E} \to \mathbb{E}$. The three statements below describe the same property:*

(1) A is order LWDC.
(2) For each disjoint sequence (ψ_n) in the unit ball $U_{\mathbb{E}^*}$, $|\psi_n - A^*(\psi_n)| \to 0$ in $\sigma(\mathbb{E}^*, \mathbb{E})$.
(3) For every order bounded sequence (u_n) in $\mathbb{E}$ and disjoint sequence (ψ_n) in $U_{\mathbb{E}^*}$, we have $\psi_n(u_n - Au_n) \to 0$.

Proof First, (1) $\Rightarrow$ (2): Assume that A is order LWDC and let (u_n) be an order bounded sequence in $\mathbb{E}$. By definition, the set $M = u_n - Au_n : n \in \mathbb{N}$ is LW-compact. By Lemma 3.2, every disjoint sequence (ψ_n) in $U_{\mathbb{E}^*}$ converges uniformly to zero on M, that is,

$$\sup_{x \in M} |\psi_n(x)| \longrightarrow 0.$$

Since $(\psi_n - A^*\psi_n)(u_n) = \psi_n(u_n - Au_n)$, we obtain

$$|(\psi_n - A^*\psi_n)(u_n)| \le \sup_{x \in M} |\psi_n(x)| \longrightarrow 0.$$

Consequently, $|\psi_n - A^*(\psi_n)| \xrightarrow{\sigma(\mathbb{E}^*,\mathbb{E})} 0$.

Next, (2) $\Rightarrow$ (3): Take disjoint (ψ_n) in $U_{\mathbb{E}^*}$ and order bounded (u_n) in $\mathbb{E}^+$. For $w_n \in [0, u_n]$, we get $|\psi_n(w_n - Aw_n)| \leq |\psi_n - A^*\psi_n|(w_n) \to 0$. So $\psi_n(u_n - Au_n) \to 0$.

Last, (3) $\Rightarrow$ (2): Take disjoint (ψ_n) in $U_{\mathbb{E}^*}$. Suppose $|\psi_n - A^*\psi_n|$ does not go to zero in $\sigma(\mathbb{E}^*, \mathbb{E})$. Therefore, there is $x \in \mathbb{E}^+$ with $|\psi_n - A^*\psi_n|(x) \not\to 0$. So we can find $\varepsilon > 0$ and $(\psi_{\phi(n)})$ such that $|\psi_{\phi(n)} - A^*\psi_{\phi(n)}|(x) > \varepsilon$. For each n, there is $u_{\phi(n)} \in [0, x]$ with $|\psi_{\phi(n)}(u_{\phi(n)} - Au_{\phi(n)})| > \varepsilon$. But condition (3) says this goes to 0. This is impossible. So $|\psi_n - A^*\psi_n| \to 0$ in $\sigma(\mathbb{E}^*, \mathbb{E})$. □

By applying the same argument, we get the next result.

Theorem 3.2 *Let $S : \mathbb{E}^* \to \mathbb{E}^*$. Then, S is order LWDC exactly when $(h_n - S(h_n))(u_n) \to 0$ for all order bounded sequences (h_n) in $\mathbb{E}^*$ and all disjoint sequences (u_n) in $U_{\mathbb{E}}$ (the unit ball of $\mathbb{E}$).*

Consequently, we get important descriptions for Banach lattices. We can use it to describe when the norm of the lattice or the dual norm is order continuous.

Corollary 3.1 *The following conditions are the same:*

(1) $I_{\mathbb{E}}$ is order LWDC.
*(2) $\mathbb{E}$ has an **order continuous norm**.*
(3) If (g_n) is any disjoint sequence in the unit ball $U_{\mathbb{E}^*}$, then $|g_n| \to 0$ in $\sigma(\mathbb{E}^*, \mathbb{E})$.
(4) If (y_n) is order bounded and (g_n) is disjoint in $U_{\mathbb{E}^*}$, then $g_n(y_n) \to 0$.

Corollary 3.2 *The three statements below have the same meaning:*

(1) $I_{\mathbb{E}^*}$ is order LWDC.
(2) The norm of $\mathbb{E}^*$ is order continuous.
(3) Whenever (g_n) is order bounded in $\mathbb{E}^*$ and (y_n) is disjoint in $U_{\mathbb{E}}$, we get $g_n(y_n) \to 0$.

Definition 3.2 $A : \mathbb{E} \to \mathbb{E}$ is named order M-weakly demicompact (for short MWDC) if the following condition holds: For any pairwise disjoint sequence (u_n) from $U_{\mathbb{E}}$ and any order bounded sequence (f_n) from the dual space of $\mathbb{E}$, whenever $f_n(u_n - Au_n) \to 0$, then necessarily $f_n(u_n) \to 0$.

Proposition 3.5 *Every order MW-compact operator $A : \mathbb{E} \to \mathbb{E}$ is also order MWDC.*

Proof Let (u_n) be pairwise disjoint elements in the unit ball of $\mathbb{E}$ and let (f_n) be an order bounded sequence in the dual of $\mathbb{E}$. Suppose $f_n(u_n - Au_n) \to 0$. Since A is order MW-compact, $f_n(Au_n) \to 0$. Observe that $f_n(u_n) \leq f_n(u_n - Au_n) + f_n(Au_n)$. Both added terms converge to zero. So $f_n(u_n) \to 0$. □

Remark 3.3 We see that every operator that is MW-compact must be order MWDC. This holds because, if (z_n) is a sequence in $\mathbb{E}$ with norms going to zero, then any order bounded sequence (f_n) in $\mathbb{E}^*$ satisfies $f_n(z_n) \to 0$. The converse fails in many cases. As a counterexample, look at the identity map I_{ℓ_∞}. The space ℓ_∞ is the dual of ℓ_1, and its norm is order continuous. Using Corollary 3.2, I_{ℓ_∞} is order MWDC. However, I_{ℓ_1} is not LW-compact. Hence, its adjoint I_{ℓ_∞} cannot be MW-compact.

$A : \mathbb{E} \to \mathbb{E}$ is power order MW-compact, if there is an integer $m \geq 0$ with the iterate A^m which becomes order MW-compact. The result below extends Proposition 3.5.

Proposition 3.6 *Let $A : \mathbb{E} \to \mathbb{E}$ be a power order MW-compact, and then it is also order MWDC.*

Proof Take a disjoint sequence (u_n) in the unit ball of $\mathbb{E}$ and an order bounded sequence (f_n) in the dual $\mathbb{E}^*$. Suppose $f_n(u_n - Au_n) \to 0$. Our goal is to show $f_n(u_n) \to 0$. Since A has the power order MW-compact property, there is $m \in \mathbb{N}^*$ such that A^m is order MW-compact. Hence, $f_n(A^m u_n) \to 0$. We can express

$$f_n(u_n) \leq f_n(u_n - A^m u_n) + f_n(A^m u_n).$$

Note that

$$u_n - A^m u_n = (I + A + A^2 + \cdots + A^{m-1})(u_n - Au_n).$$

Because $f_n(u_n - Au_n) \to 0$ and the sum $I + A + \cdots + A^{m-1}$ is a fixed linear operator, $f_n(u_n - A^m u_n) \to 0$. Since both extra terms converge to zero, we conclude that $f_n(u_n) \to 0$. This completes the argument. □

Theorem 3.3 *Let $\mathbb{E}$ be the finite direct sum of Banach lattices: $\mathbb{E} = \bigoplus_{k=1}^{m} \mathbb{E}_k$. For each pair (i, j) with $1 \leq i, j \leq m$, let $B_{i,j} : \mathbb{E}_j \to \mathbb{E}_i$ be a linear operator.*

Suppose we have the following:

(1) Each diagonal operator $B_{i,i} : \mathbb{E}_i \to \mathbb{E}_i$ is order MWDC ($i = 1, \ldots, p$).

(2) Each upper triangular operator $B_i^j : \mathbb{E}_j \to \mathbb{E}_i$ is order MW-compact when $i < j$.

Then, the associated matrix operator $\mathcal{B} : \mathbb{E} \to \mathbb{E}$, given in block form by

$$\mathcal{B} = \left(B_i^j\right)_{1 \leq i,j \leq p} = \begin{pmatrix} B_1^1 & B_1^2 & \cdots & B_1^p \\ B_2^1 & B_2^2 & \cdots & B_2^p \\ \vdots & \vdots & \ddots & \vdots \\ B_p^1 & B_p^2 & \cdots & B_p^p \end{pmatrix},$$

is order MWDC.

Proof Consider a disjoint sequence (u_n) belonging to the unit ball of $\mathbb{E}$. For each n, we have $u_n = (u_n^1, u_n^2, \ldots, u_n^p)$ where $u_n^i \in E_i$. Let an order bounded sequence $(f_{n,\mathbb{E}})$ in the dual $\mathbb{E}^*$ be such that $f_{n,\mathbb{E}}(u_n - \mathcal{B}u_n) \to 0$. We must show that $f_{n,\mathbb{E}}(u_n) \to 0$. Since

$$f_{n,\mathbb{E}}(u_n) = f_{n,\mathbb{E}_1}(u_n^1) + f_{n,\mathbb{E}_2}(u_n^2) + \cdots + f_{n,\mathbb{E}_p}(u_n^p),$$

it suffices to prove that $f_{n,\mathbb{E}_i}(u_n^i) \to 0$ for each $1 \leq i \leq p$. For every n, we have

$$u_n - \mathcal{B}u_n = \begin{pmatrix} u_n^1 - B_1^1 u_n^1 - B_1^2 u_n^2 - \cdots - B_1^p u_n^p \\ u_n^2 - B_2^1 u_n^1 - B_2^2 u_n^2 - \cdots - B_2^p u_n^p \\ \vdots \\ u_n^p - B_p^1 u_n^1 - B_p^2 u_n^2 - \cdots - B_p^p u_n^p \end{pmatrix}.$$

Hence, $f_{n,E_i}\left(u_n^i - B_i^1 u_n^1 - B_i^2 u_n^2 - \cdots - B_i^p u_n^p\right) \to 0, \quad \forall\, 1 \leq i \leq p.$

Case $i = 1$. Since (u_n^j) is disjoint and each B_1^j $(j \geq 2)$ is order MW-compact, we have $f_{n,\mathbb{E}_1}(B_1^j u_n^j) \to 0, \quad \forall\, 2 \leq j \leq p$. From the inequality

$$f_{n,\mathbb{E}_1}(u_n^1 - B_1^1 u_n^1) \leq f_{n,\mathbb{E}_1}\left(u_n^1 - B_1^1 u_n^1 - B_1^2 u_n^2 - \cdots - B_1^p u_n^p\right) + \sum_{j=2}^{p} f_{n,\mathbb{E}_1}(B_1^j u_n^j),$$

we deduce that $f_{n,\mathbb{E}_1}(u_n^1 - B_1^1 u_n^1) \to 0$. Since B_1^1 is order MWDC, it follows that $f_{n,\mathbb{E}_1}(u_n^1) \to 0$.

Case $2 \leq i \leq p - 1$. For these indices, the order MW-compactness of each $B_{i,j}$ implies $f_{n,\mathbb{E}_i}(B_i^j x_n^j) \to 0, \quad \forall\, i < j \leq m$. Using the inequality

$$\begin{aligned} f_{n,\mathbb{E}_i}(u_n^i - B_i^i u_n^i) \leq\ & f_{n,\mathbb{E}_i}\left(u_n^i - B_i^1 u_n^1 - \cdots - B_i^{i-1} u_n^{i-1} - B_i^{i+1} u_n^{i+1} - \cdots - B_i^p u_n^p\right) \\ & + \sum_{j=1}^{i-1} f_{n,\mathbb{E}_i}(B_i^j u_n^j) + \sum_{j=i+1}^{p} f_{n,\mathbb{E}_i}(B_i^j u_n^j), \end{aligned}$$

we conclude that $f_{n,\mathbb{E}_i}(u_n^i - B_i^i u_n^i) \to 0$. Since B_i^i is order MWDC, this yields $f_{n,\mathbb{E}_i}(u_n^i) \to 0$.

Case $i = p$. We have

$$f_{n,\mathbb{E}_p}(u_n^p - B_p^p u_n^p) \leq f_{n,\mathbb{E}_p}\left(u_n^p - B_p^1 u_n^1 - B_p^2 u_n^2 - \cdots - B_p^p u_n^p\right) + \sum_{j=1}^{p-1} f_{n,\mathbb{E}_p}(B_p^j u_n^j).$$

Thus, $f_{n,\mathbb{E}_p}(u_n^p - B_p^p u_n^p) \to 0$, and the order MWDC of B_p^p implies $f_{n,\mathbb{E}_p}(u_n^p) \to 0$.

Combining all the above cases, we conclude that $f_{n,\mathbb{E}}(u_n) \to 0$. Therefore, $\mathcal{B}$ is order MWDC. □

Lemma 3.3 *Let $A : \mathbb{E} \to \mathbb{E}$ be order MWDC, and let $S : \mathbb{E} \to \mathbb{E}$ be order MW-compact. Then, $A + S$ is order MWDC.*

Proof Consider a disjoint sequence (u_n) in the unit ball of $\mathbb{E}$ and an order bounded sequence (f_n) in the dual $\mathbb{E}^*$. Assume that $f_n(u_n - (A + S)u_n) \to 0$ as $n \to \infty$. The property of S (order MW-compact) gives $f_n(Su_n) \to 0$ as $n \to \infty$. Notice that $f_n(u_n - Au_n) \leq |f_n(u_n - (A + S)u_n)| + |f_n(Su_n)|$. The right-hand side tends to zero. Therefore, $f_n(u_n - Au_n) \to 0$. By the definition of order MWDC for A, we get $f_n(u_n) \to 0$. Hence, $A + S$ is order MWDC. □

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