Research Article

Imen Ferjani, Omaima Kchaou and Bilel Krichen*

Generalized essential spectra involving the class of g-g-Riesz operators

https://doi.org/10.1515/gmj-2024-2002
Received May 27, 2023; revised September 11, 2023; accepted September 21, 2023

Abstract: In this paper, we explore the spectral properties of unbounded generalized Fredholm operators acting on a non-reflexive Banach space X. The results are formulated in terms of some topological conditions made on X or on its dual X^*. In addition, we introduce the concept of the so-called g-g-Riesz linear operators as an extension of Riesz operators. The obtained results are used to discuss the incidence of the behavior of generalized essential spectra. Furthermore, a relation between the generalized essential spectrum and the left (resp. the right) essential spectrum by means of g-Riesz perturbation is provided.

Keywords: g-Riesz operators, g-g-Riesz operators, generalized Fredholm operators, generalized essential spectrum

MSC 2020: 47A53, 47A10

1 Introduction

In 1954, A. F. Ruston [17] raised the concept of Riesz-type operators in terms of some spectral properties of compact operators which has been given by F. Riesz in his study of integral equations. In 1966, S. R. Caradus [10] showed that the classes of Riesz and operators whose Fredholm region consists of all non-zero complex numbers coincide, where the Fredholm region is defined by the set $\Phi_T = \{\lambda : \lambda - T \text{ is a Fredholm operator}\}$. An investigation of connection between Riesz operators and Fredholm perturbations has been given by M. Schechter [18] as a complement of [4, 10, 16] and others papers [2, 3]. In 2006, the author of [15] introduced the concept of generalized Riesz operators and described the Riesz-Schauder theory of compact operators in the more general setting of polynomially compact operators.

As is well known, all characterizations of Fredholm operators are developed in terms of invertibility modulo compact set operators. Sadly, in practical application we face various problems associated with the loss of compactness of the operators concerned. For this reason, it is convenient that we put our focus on a larger class of Fredholm operators in non-reflexive Banach spaces. Accordingly, in 1976, K. W. Yang [21] initiated the class of generalized Fredholm operators, and provided a characterization using weakly compact sets. Recently, in [6], A. Azzouz, M. Beghdadi and B. Krichen extended some of the results of [21] and gave some sufficient conditions to an operator to be generalized semi-Fredholm. Their results were formulated in terms of some quantities related with measures of weak noncompactness. However, all these studies were restricted to bounded linear operators and missed perturbation results. For more details, we need to introduce some standard notation from Fredholm theory. Let X and Y be two Banach spaces. We define by $\mathcal{C}(X, Y)$ the set of all closed linear operators from X into Y having dense domains and by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X into Y. If $X = Y$, then $\mathcal{C}(X, Y)$ and $\mathcal{L}(X, Y)$ are respectively replaced by $\mathcal{C}(X)$ and $\mathcal{L}(X)$. For $G \in \mathcal{C}(X, Y)$, the sets $\Theta(G)$, $\mathcal{R}(G)$, $Y/\mathcal{R}(G)$ and $\mathcal{N}(G)$ respectively refer to the domain, the range, the co-kernel and the kernel of G. If $G \in \mathcal{C}(X, Y)$, then $\Theta(G)$

***Corresponding author: Bilel Krichen,** Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Road of Soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia, e-mail: krichen_bilel@yahoo.fr. https://orcid.org/0000-0002-3481-7575
Imen Ferjani, Omaima Kchaou, Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Road of Soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia, e-mail: imenoferjani@yahoo.fr, kchaouomaima53@gmail.com

endowed with the graph norm $\|x\|_G = \|x\| + \|Gx\|$ is a Banach space denoted by X_G. Note that the restriction $\widehat{G}$ of the closed operator G on $\Theta(G)$ is bounded from X_G into Y. If $\Theta(G) \subset \Theta(V)$, then V will be called G-defined. So, the restriction of V to X_G is denoted by $\widehat{V}$. The linear space X^* denote the topological dual of the Banach space X.

Now, we recall that an operator $K \in \mathcal{L}(X, Y)$ is said weakly compact if $K(M)$ is relatively weakly compact in Y for every bounded subset $M \subset X$. The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X, Y)$. If $X = Y$, then $\mathcal{W}(X, X) := \mathcal{W}(X)$. If $T \in \mathcal{C}(X)$ and M is a relatively weakly compact subset of X, then M is not necessarily weakly compact in X_T. However, in [1], it was proved under a certain density condition that if both M and $T(M)$ are relatively weakly compact sets of X, then M is a relatively weakly compact set of X_T. For some extensions to the unbounded case, we make use of the following assumption for $T \in \mathcal{C}(X)$:

$$X^* + X^* \circ T \text{ is dense in } (X_T)^*. \tag{$\mathcal{P}_1$}$$

In what follows, we will use the notation given in [6] and extend them to unbounded setting. The classes of upper generalized semi-Fredholm operators, lower generalized semi-Fredholm operators and generalized Fredholm operators are respectively denoted by

$$\Phi_{g+}(X, Y) := \{T \in \mathcal{C}(X, Y) : \mathcal{N}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\},$$
$$\Phi_{g-}(X, Y) := \{T \in \mathcal{C}(X, Y) : Y/\mathcal{R}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\}$$

and

$$\Phi_g(X, Y) := \Phi_{g+}(X, Y) \cap \Phi_{g-}(X, Y).$$

When $X = Y$, all the sets $\Phi_g(X, Y)$, $\Phi_{g+}(X, Y)$ and $\Phi_{g-}(X, Y)$ are simply substituted by $\Phi_g(X)$, $\Phi_{g+}(X)$ and $\Phi_{g-}(X)$, respectively. Let us consider a complex number $\lambda \in \Phi_{g,T}(X, Y)$ if $\lambda - T \in \Phi_g(X, Y)$. Additionally, we note the obvious relations

$$\begin{cases} \mathcal{N}(\widehat{G}) = \mathcal{N}(G), & \mathcal{N}(\widehat{G} + \widehat{J}) = \mathcal{N}(G + J), \\ \mathcal{R}(\widehat{G}) = \mathcal{R}(G), & \mathcal{R}(\widehat{G} + \widehat{J}) = \mathcal{R}(G + J). \end{cases}$$

Then we can see that $G \in \Phi_g(X)$ if and only if $\widehat{G} \in \Phi_g(X_G, X)$.

For $F \in \mathcal{C}(X, Y)$, we define the resolvent set of F by

$$\varrho(F) := \{\lambda \in \mathbb{C} : \lambda - F \text{ has a bounded inverse}\},$$

and the spectrum of F by

$$\sigma(F) := \mathbb{C} \setminus \varrho(F).$$

Several definitions of the essential spectrum may be found in the literature (see [7, 16, 18]). In this work, we are interested in the generalized Wolf essential spectrum and the generalized Gustafson essential spectrum of a closed densely defined linear operator T. These essential spectra are respectively defined by

$$\sigma_{e_1,g}(T) := \{\lambda \in \mathbb{C} : (\lambda - T) \notin \Phi_{g+}(X)\},$$
$$\sigma_{e_4,g}(T) := \{\lambda \in \mathbb{C} : (\lambda - T) \notin \Phi_g(X)\}.$$

The generalized resolvent $\varrho_g(T)$ is $\mathbb{C} \setminus \sigma_{e_4,g}(T)$.

The purpose of this paper is the study of the class of the so-called g-g-Riesz linear operators acting on Banach spaces (see Section 4), in particular on spaces satisfying properties (H_1) and (H_2) (see Definition 2.5) and its various beneficial interactions in the spectral theory. In particular, we will extend some results of generalized Fredholm operators developed in [6] to the case of unbounded linear operators acting on a Banach space using assumption ($\mathcal{P}_1$). These results are applied to study the class of g-g-Riesz of linear operators acting on a Banach space and to establish the stability of various generalized essential spectra. We finish by giving a relation between the generalized essential spectrum and the left (resp. the right) essential spectrum by means of generalized Riesz perturbation.

The paper can be outlined as follows. In Section 2, we recall some definitions and results that will be utilized throughout the rest of the paper. In Section 3, we introduce some results concerning the class of generalized Fredholm operators with respect to a given unbounded linear operator. In Section 4, we use the results obtained in Section 3 to investigate the class of g-g-Riesz operators and study some stability results for generalized essential spectra.

2 Preliminary results

We initiate this section by introducing some notations and useful definitions. Let X be a Banach space and $B_r = B(0, r)$ be the open ball of X centered at 0_X and with radius $r > 0$. We denote by $\mathcal{M}_X$ (resp. $\mathcal{M}_{X_T}$) the set of bounded subsets of X (resp. X_T), where $T \in \mathcal{C}(X)$, and by $\mathcal{W}_X$ the set of all weakly compact subsets of X and by $\mathcal{K}_X$ the set of all relatively weakly compact subsets of X. Moreover, let conv(A) denote the convex hull of the set $A \subset X$. Note that $\bar{A}^{\Omega}$ is the weak closure of A in X.

The notion of a weak noncompactness measure was first introduced by F. S. De Blasi [12] and has found applications in topology and functional analysis (see [4, 9, 13]). An axiomatic definition was given in [8, 9] as follows.

Definition 2.1. Let X be a Banach space. A function $\upsilon : \mathcal{M}_X \to [0, +\infty[$ is said to be a measure of weak noncompactness in X (in short, MWNC) if for all $A, B \in \mathcal{M}_X$, it satisfies the following conditions:
(i) $\mathcal{N}(\upsilon) = \{A \in \mathcal{M}_X$ such that $\upsilon(A) = 0\}$ is nonempty and $\mathcal{N}(\upsilon) \subset \mathcal{K}_X$.
(ii) If $A \subset B$, then $\upsilon(A) \leq \upsilon(B)$.
(iii) $\upsilon(\overline{\mathrm{conv}(A)}) = \upsilon(A)$.
(iv) $\upsilon(\lambda A + (1 - \varsigma)B) \leq \varsigma\upsilon(A) + (1 - \varsigma)\upsilon(B)$ for $\varsigma \in [0, 1]$.
(v) If $A_n \in \mathcal{M}_X$ are such that A_n is weakly closed in X with $A_{n+1} \subset A_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to+\infty} \upsilon(A_n) = 0$, then $A_\infty := \bigcap_{i=1}^{+\infty} A_n \neq \emptyset$.

Definition 2.2 ([8, 9]). Let X be a Banach space and υ be an MWNC in X. Then:
(i) υ satisfies the maximum property if $\upsilon(A \cup B) = \max\{\upsilon(A), \upsilon(B)\}$ for any $A, B \in \mathcal{M}_X$.
(ii) υ is called homogeneous if $\upsilon(\varsigma A) = |\varsigma|\upsilon(A)$ for any $A \in \mathcal{M}_X$ and $\varsigma \in \mathbb{C}$.
(iii) υ is called subadditive if $\upsilon(A + B) \leq \upsilon(A) + \upsilon(B)$ for any $A, B \in \mathcal{M}_X$.
(iv) υ is called sublinear if it is homogeneous and subadditive.
(v) υ is called regular if it is sublinear, has the maximum property and $\mathcal{N}(\upsilon) = \mathcal{K}_X$.

As an example of a measure of weak noncompactness in a Banach space, we cite the De Blasi measure Ω defined on $\mathcal{M}_X$ as follows:

$$\Omega(A) = \inf\{r > 0 : \text{there exists } N \in \mathcal{W}_X \text{ such that } A \subset N + \overline{B}_r\}.$$

Definition 2.3. Let X be a Banach space and let $T \in \mathcal{L}(X)$. We define the measure of weak noncompactness of T, denoted by $\varpi(T)$, as follows:

$$\varpi(T) = \inf\{k > 0 : \Omega(T(A)) \leq k\Omega(A) \text{ for all } A \in \mathcal{M}_X\}.$$

Proposition 2.1 ([6]). *Let X be a Banach space, let $T, S \in \mathcal{L}(X)$ and let $B \in \mathcal{M}_X$. Then we have the following properties:*
(i) $\varpi(T) = 0$ *if and only if T is weakly compact,*
(ii) $\Omega(T(B)) \leq \varpi(T)\Omega(B)$,
(iii) $\varpi(TS) \leq \varpi(T)\varpi(S)$,
(iv) $\varpi(T + S) \leq \varpi(T) + \varpi(S)$,
(v) $\varpi(\lambda T) = |\lambda|\varpi(T)$ *for* $\lambda \in \mathbb{C}$.

In the sequel, we use the following definition.

Definition 2.4. Let X be a Banach space and let $T \in \mathcal{C}(X)$. We define the measure of weak noncompactness of T, denoted by $\tilde{\upsilon}(T)$, as follows:

$$\tilde{\upsilon}(T) = \inf\{\varepsilon > 0 : \upsilon(T(A)) \leq \varepsilon\upsilon(A) \text{ for all } A \in \mathcal{M}_{X_T}\}.$$

The following properties hold.

Remark 2.1. Let X be a Banach space, $T \in \mathcal{C}(X)$, υ an MWNC in X and let $A \in \mathcal{M}_X$. Then $\upsilon(T(A)) \leq \tilde{\upsilon}(T)\upsilon(A)$.

Proof. The proof follows from Definition 2.4. □

Definition 2.5. Let X be a Banach space. We say that X has property (H_1) (resp. (H_2)) if every closed reflexive subspace admits a closed complementary subspace (resp. if every closed subspace with reflexive quotient space admits a closed complementary subspace). We say that X has property (H) if it satisfies both properties (H_1) and (H_2).

Note that if $1 < p < \infty$, then $L_p(0,1)$ has property (H_1). Moreover, $L_1(0,1)$ and $\mathcal{C}(S)$, the space of all bounded (real or complex valued) continuous functions on an infinite compact Hausdorff space S, do not have it [14].

Theorem 2.1 ([21]). *Let X and Y be two Banach spaces satisfying properties (H_1) and (H_2), respectively, and let $T \in \mathcal{L}(X,Y)$. Then the following assertions are equivalent:*
(i) *T is a generalized Fredholm operator.*
(ii) *There exist weakly compact operators $W_1 \in \mathcal{W}(X)$ and $W_2 \in \mathcal{W}(Y)$ and an operator $T_0 \in \mathcal{L}(Y,X)$ such that $T_0T = I + W_1$, $TT_0 = I + W_2$ and $\mathcal{R}(T)$ is closed.*

Theorem 2.2 ([6]). *Let X be a Banach space and let $T \in \mathcal{L}(X)$ such that $\varpi(T^n) < 1$ for some $n \in \mathbb{N} \setminus \{0\}$. Then $(I - T) \in \Phi_g(X)$.*

Lemma 2.1 ([1]). *Let X be a Banach space and let $T \in \mathcal{C}(X)$. For $A \in \mathcal{M}_X$ (resp. $A \in \mathcal{M}_{X_T}$), we denote by $\overline{A}^{\Omega}$ (resp. $\overline{A}^{\Omega T}$) the weak closure of A in X (resp. in X_T). Then:*
(i) *$X^* + X^* \circ T \subset (X_T)^*$.*
(ii) *If $x \in \overline{A}^{\Omega T}$, then $x \in \overline{A}^{\Omega}$ and $T(x) \in \overline{T(A)}^{\Omega}$.*
(iii) *Moreover, if we suppose that $X^* + X^* \circ T$ is dense in $(X_T)^*$, then for any $A \in \mathcal{M}_{X_T}$ we have*

$$A, T(A) \in \mathcal{W}_X \implies A \in \mathcal{W}_{X_T}.$$

(iv) *If A is weakly compact in X_T, then A is weakly compact in X.*

Lemma 2.2 ([5]). *Let X be a non-reflexive Banach space satisfying property (H_1) and $T \in \mathcal{L}(X)$. If T is left and right invertible modulo the weakly compact operators, then $T \in \Phi_g(X)$.*

3 Generalized Fredholm theory by means of the graph measure of weak noncompactness

Let X be a Banach space, let $T \in \mathcal{C}(X)$ and let υ be an MWNC in X. We define

$$\mathcal{H}_\upsilon(X) := \Big\{T \in \mathcal{C}(X) : A_n \in \mathcal{M}_{X_T} \text{ weakly closed in } X_T \text{ with } A_{n+1} \subset A_n, \\ \text{if } \lim_{n\to+\infty} \upsilon(A_n) = 0 \text{ and } \lim_{n\to+\infty} \upsilon(T(A_n)) = 0, \text{ then } \bigcap_{i=0}^{+\infty} A_n \neq \emptyset\Big\}$$

and consider the following non-negative quantities:

$$\gamma(T) := \sup\left\{\frac{\upsilon(T(A))}{\upsilon(A)} : A \in \mathcal{M}_{X_T} \text{ and } \upsilon(A) > 0\right\} = \tilde{\upsilon}(T),$$
$$\delta(T) := \inf\left\{\frac{\upsilon(T(A))}{\upsilon(A)} : A \in \mathcal{M}_{X_T} \text{ and } \upsilon(A) > 0\right\}.$$

Theorem 3.1 ([1]). *Let X be a Banach space and let υ be an MWNC in X. Let $T \in \mathcal{H}_\upsilon(X)$. We define $\upsilon_T : \mathcal{M}_{X_T} \to \mathbb{R}^+$ by*

$$\upsilon_T(A) = \upsilon(A) + \upsilon(T(A)) \quad \text{for all } A \in \mathcal{M}_{X_T}.$$

Assume that υ is homogeneous. If we suppose that $X^* + X^* \circ T$ is dense in $(X_T)^*$, then υ_T is a measure of weak noncompactness in X_T called a graph measure of weak noncompactness associated to υ and T.

Now, we give some properties of γ and δ which will be useful in the sequel.

Proposition 3.1. *Let X be a Banach space, let $T \in \mathcal{C}(X)$, let υ be an MWNC in X and let $\alpha \in \mathbb{C}$. Then the following statements hold:*

(i) *If υ is homogeneous, then $\gamma(\alpha T) = |\alpha|\gamma(T)$ and $\delta(\alpha T) = |\alpha|\delta(T)$.*

(ii) *If υ is regular, then $\gamma(T) = 0$ if and only if T is weakly compact.*

(iii) *If υ is regular, then $\delta(T + W) = \delta(T)$ for all $W \in \mathcal{W}(X)$.*

Proof. Statements (i) and (iii) follow from the definitions of γ and δ together with Definition 2.2.

(ii) Let A be a bounded set of X_T such that $\upsilon(A) > 0$. Taking into account that $\gamma(T) = 0$, it follows that $\frac{\upsilon(T(A))}{\upsilon(A)} = 0$, hence $T(A)$ is relatively weakly compact. To prove the converse, since $T(A)$ is relatively weakly compact, we infer that $\frac{\upsilon(T(A))}{\upsilon(A)} = 0$, and so $\gamma(T) = 0$. □

Definition 3.1. Let X and Y be two Banach spaces and let $T \in \mathcal{C}(X, Y)$. We say that T is weakly proper if for every $W \in \mathcal{W}_Y$, $T^{-1}(W) \in \mathcal{W}_X$.

Example 3.1. Let ν be a bounded domain of $\mathbb{R}^n$, $n = 2, 3$. We define the Navier–Stokes operator (in bounded domain) by

$$N : (0, \infty) \times \mathcal{D}_0^{1,2}(\nu) \to \mathcal{D}_0^{1,2}(\nu), \quad (\vartheta, u) \mapsto \vartheta u - \mathcal{N}(u),$$

where $\mathcal{D}_0^{1,2}(\nu)$ is called Leray class. Clearly, $\mathcal{D}(N) = \mathcal{D}_0^{1,2}(\nu)$ and $\mathcal{R}(N) = \mathcal{D}_0^{1,2}(\nu)$. The Navier–Stokes operator N is uniformly coercive in $\vartheta > 0$. So, by [11, Lemma 29], it follows that N is weakly proper at every $F \in \mathcal{D}_0^{1,2}(\nu)$.

In what follows, let X and X_T be two Banach spaces and $T \in \mathcal{H}_\upsilon(X)$, we make the following assumption:

$$X^* + X^* \circ T \text{ is dense in } (X_T)^*. \tag{$\mathcal{P}_1$}$$

Lemma 3.1. *Let X be a Banach space and let υ be a regular MWNC in X. Let $T \in \mathcal{H}_\upsilon(X)$ satisfy hypothesis $(\mathcal{P}_1)$. If $\delta(T) > 0$, then T is weakly proper on bounded subset of X_T.*

Proof. Let W be a weakly compact set of X_T and let D be a weakly closed bounded subset of X_T. It remains to prove that $M = D \cap T^{-1}(W)$ is relatively weakly compact in X_T. Clearly, $M \in \mathcal{M}_{X_T}$. Assuming the contrary, we have $\upsilon_T(M) > 0$.

Case 1: Suppose that $\upsilon(T(M)) > 0$. We have

$$T(M) = T(D \cap T^{-1}(W)) \subset T(D) \cap T(T^{-1}(W)) \subset T(D) \cap W,$$

this mean that

$$\upsilon(T(M)) \leq \upsilon(T(D) \cap W) \leq \upsilon(W).$$

Hence $\upsilon(T(M)) = 0$, which is a contradiction.

Case 2: Assume that $\upsilon(M) > 0$. Taking into account that $\delta(T)\upsilon(M) \leq \upsilon(T(M))$, we have $\delta(T) = 0$. Contradiction. Hence $\upsilon(T(M)) > 0$ and $\upsilon(M) > 0$. Therefore, we obtain that $\overline{M}^{\Omega}$ and $\overline{T(M)}^{\Omega}$ are weakly compact on X. Now, according to Lemma 2.1, we conclude that M is relatively weakly compact in X_T. This proves the result. □

Proposition 3.2. *Let X be a non-reflexive Banach space and let υ be a regular MWNC in X. Let $T \in \mathcal{H}_\upsilon(X)$ satisfy hypothesis $(\mathcal{P}_1)$. If $\delta(T) > 0$, then T is an upper generalized Fredholm operator.*

Proof. Taking into account that $\delta(T) > 0$, it follows from Lemma 3.1 that T is weakly proper on bounded subsets of X_T. Then $\widehat{T}$ is weakly proper on bounded subsets of X. Taking into account that $B \cap \widehat{T}^{-1}(W) \in \mathcal{W}(X)$ for all $B \in \mathcal{M}_X$, using [6, Theorem 3.5], we deduce that $\widehat{T} \in \Phi_{g+}(X_T, X)$. Consequently, T is an upper generalized Fredholm operator. □

Theorem 3.2 ([5]). *Let X be a Banach space having property (H_1) and let $T \in \mathcal{L}(X)$. If $T \in \Phi_{g^+}(X)$, then there exist $S \in \mathcal{L}(X)$ and $W \in \mathcal{W}(X)$ such that $ST = I - W$.*

Theorem 3.3. *Let X be a non-reflexive Banach space, let υ be a regular MWNC in X and let $T \in \mathcal{C}(X)$. Assume that X_T satisfies property (H_1) and T is a generalized Fredholm. Then $\delta(T) > 0$.*

Proof. Since $T \in \Phi_{g+}(X)$, we infer that $\widehat{T} \in \Phi_{g+}(X_T, X)$. Taking into account that X_T satisfies property (H_1), due to Theorem 3.2, there exist $F \in \mathcal{L}(X, X_T)$ and $W \in \mathcal{W}(X_T)$ such that $F\widehat{T} = I_{X_T} - W$. To prove the result, we suppose the opposite, i.e., we have $\delta(T) = 0$ and $\varepsilon < \frac{1}{\tilde{\upsilon}(F)}$. In that case there exists $D \in \mathcal{M}_{X_T}$ with $\upsilon(D) > 0$ and $\frac{\upsilon(T(D))}{\upsilon(D)} \leq \frac{1}{\tilde{\upsilon}(F)}$. Then

$$\upsilon(T(D)) \leq \varepsilon\upsilon(D) \leq \varepsilon\upsilon((F\widehat{T} + W)D) \leq \varepsilon\tilde{\upsilon}(F)\upsilon(T(D)).$$

Therefore, we have $(1 - \varepsilon\tilde{\upsilon}(F))\upsilon(T(D)) \leq 0$. Using the fact $\varepsilon\tilde{\upsilon}(F) < 1$, we infer that $\upsilon(T(D)) = 0$, which implies that $T(D)$ is relatively weakly compact. Now, it can be seen that $\upsilon(D) = \upsilon(F\widehat{T}(D) + W(D))$. Consequently, we have $\upsilon(D) = 0$, contradiction. □

Theorem 3.4. *Let X be a non-reflexive Banach space, let υ be a regular MWNC in X and let $T \in \mathcal{H}_\upsilon(X)$ satisfying hypothesis $(\mathcal{P}_1)$. Let $S \in \mathcal{L}(X)$ and let X_T satisfy property (H_1). Then from $ST \in \Phi_{g+}(X)$, we have $T \in \Phi_{g+}(X)$.*

Proof. Let $ST \in \Phi_{g+}(X)$. Then $S\widehat{T} \in \Phi_{g+}(X_T, X)$. Since X_T satisfies property (H_1), there exist $K \in \mathcal{L}(X, X_T)$ and $W_1 \in \mathcal{W}(X_T)$ such that $KS\widehat{T} = I_{X_T} + W_1$. Arguing as in the proof of Theorem 3.3, we deduce that $\delta(T) > 0$. Now, applying Proposition 3.2, we infer that $T \in \Phi_{g+}(X)$. □

Remark 3.1. Let X be a Banach space, let $T \in \mathcal{C}(X)$, $S \in \mathcal{L}(X)$ and let υ be an MWNC in X. Then $\delta(ST) \geq \delta(S)\delta(T)$.

Proof. As a direct consequence of the properties of υ, we obtain that

$$\frac{\upsilon(ST(A))}{\upsilon(A)} \leq \frac{\upsilon(S(T(A)))}{\upsilon(A)} \leq \gamma(S)\frac{\upsilon(T(A))}{\upsilon(A)}.$$

Then

$$\inf\left\{\frac{\upsilon(ST(A))}{\upsilon(A)} : \upsilon(A) > 0\right\} \geq \gamma(S)\inf\left\{\frac{\upsilon(T(A))}{\upsilon(A)} : \upsilon(A) > 0\right\}.$$

Hence $\delta(ST) \geq \delta(S)\delta(T)$, which completes the proof. □

Theorem 3.5. *Let X be a non-reflexive Banach space, let υ be a regular MWNC in X and let $T \in \mathcal{H}_\upsilon(X)$ satisfying hypothesis $(\mathcal{P}_1)$. Let $S \in \mathcal{L}(X)$ and X_T satisfy property (H_1). The following statements hold:*
(i) *If $T \in \Phi_{g+}(X)$ and $S \in \Phi_{g+}(X)$, then $ST \in \Phi_{g+}(X)$.*
(ii) *If $T \in \Phi_{g+}(X)$ and $W \in \mathcal{W}(X)$, then $(T + W) \in \Phi_{g+}(X)$.*

Proof. (i) Taking into account that $T, S \in \Phi_{g+}(X)$, by using Theorem 3.3, we infer that $\delta(T) > 0$ and $\delta(S) > 0$. It follows by Remark 3.1 that $\delta(ST) > 0$. Consequently, $ST \in \Phi_{g+}(X)$.

(ii) Since $T \in \Phi_{g+}(X)$, $W \in \mathcal{W}(X)$ and X_T has property (H_1), from Proposition 3.1, we get $\delta(T + W) = \delta(T) > 0$. This implies, using Proposition 3.2, that $(T + W) \in \Phi_{g+}(X)$. □

Theorem 3.6. *Let X be a non-reflexive Banach space. Let $T \in \mathcal{C}(X)$, $S \in \mathcal{L}(X)$. Assume that X_T and X satisfy properties (H_1) and (H), respectively. Then the following statements hold:*
(i) *If $T \in \Phi_g(X)$ and $S \in \Phi_g(X)$, then $ST \in \Phi_g(X)$.*
(ii) *If $T \in \Phi_g(X)$ and $W \in \mathcal{W}(X)$, then $(T + W) \in \Phi_g(X)$.*

Proof. (i) Since $T \in \Phi_g(X)$ and $S \in \Phi_g(X)$, it follows that $\widehat{T} \in \Phi_g(X_T, X)$. Since X_T satisfies property (H_1) and X has property (H), using Theorem 2.1, we infer that there exist $T_0 \in \mathcal{L}(X, X_T)$ and $W_1 \in \mathcal{W}(X_T)$, $W_2, W_4 \in \mathcal{W}(X)$ and $W_3 \in \mathcal{W}(X)$ such that $T_0\widehat{T} = I_{X_T} + W_1$, $\widehat{T}T_0 = I_X + W_2$, $S_0S = I_X + W_3$ and $SS_0 = I_X + W_4$. Then

$$T_0S_0S\widehat{T} = T_0(I_X + W_3)\widehat{T} = I_{X_T} + W_1 + T_0W_3\widehat{T} = I_{X_T} + W_5, \quad W_5 \in \mathcal{W}(X_T),$$

and

$$S\widehat{T}T_0S_0 = S(I_X + W_2)S_0 = I_X + W_4 + SW_2S_0 = I_X + W_6, \quad W_6 \in \mathcal{W}(X).$$

Using Remark 2.2, we deduce that $ST \in \Phi_g(X)$.

(ii) Arguing as in the proof of (i), we deduce that $(T + W) \in \Phi_g(X)$. □

4 The g-g-Riesz operators

In this section, we investigate the class of g-g-Riesz linear operators acting on a Banach space X having some properties.

Definition 4.1. Let X be a Banach space and let $T \in \mathcal{L}(X)$. We say that T is a g-g-Riesz operator if there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda - T \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. The set E is called a generalized set of T. We denote by $g\mathcal{R}_g(X)$ the set of g-g-Riesz operators acting on X.

Remark 4.1. We make the following observations:
(i) Every weakly compact operator is a g-g-Riesz operator.
(ii) Note that for $E = \{0\}$, we denote by $\mathcal{R}_g(X)$ the set of g-Riesz operators acting on a Banach space X (see [5]).

Theorem 4.1. *Let X be a non-reflexive Banach space having property* (H_1) *and let X^* satisfy property* (H_1). *Let $T \in \mathcal{L}(X)$ and let Q be a non-zero complex polynomial, with zeros $y_1, \dots, y_p$, such that $Q(T)$ is a g-Riesz operator. Then T is a g-g-Riesz operator with a generalized set $E = \{0, y_1, \dots, y_p\}$.*

Proof. Assume that there exists Q, a non-zero complex polynomial. Put

$$Q(y) = \sum_{j=0}^{p} a_j y^j.$$

For $\xi \in \mathbb{C} \setminus \{0, y_1, \dots, y_p\}$, we have

$$Q(\xi) - Q(T) = \sum_{j=1}^{p} a_j(\xi^j - T^j).$$

However, for any $j \in \{1, \dots, p\}$, $\xi^j - T^j = (\xi - T), \sum_{i=0}^{j-1} \xi^i T^{j-1-i}$. This allows us to write

$$Q(\xi) - Q(T) = (\xi - T)P(T) = P(T)(\xi - T), \tag{4.1}$$

where

$$P(T) = \sum_{j=1}^{p} a_j \sum_{i=0}^{j-1} \xi^i T^{j-1-i}.$$

Now, taking into account that $Q(T)$ is a g-Riesz operator, it follows that $\upsilon - Q(T) \in \Phi_g(X)$ for all $\upsilon \in \mathbb{C} \setminus \{0\}$. Then, in particular, for $\upsilon = Q(\xi)$ we get

$$Q(\xi) - Q(T) \in \Phi_g(X).$$

Using equation (4.1), we have $(\xi - T)P(T) \in \Phi_g(X)$ and $P(T)(\xi - T) \in \Phi_g(X)$. Hence, in view of [6, Theorem 3.9], we obtain that $\xi - T \in \Phi_g(X)$ for all $\xi \in \mathbb{C} \setminus \{0, y_1, \dots, y_p\}$. Consequently, T is a g-g-Riesz operator with a generalized set $E = \{0, y_1, \dots, y_p\}$. □

Theorem 4.2 ([5]). *Let X be a non-reflexive Banach space having property* (H) *and let $T, S \in \mathcal{L}(X)$ be such that $TS = ST$. Then the following statements hold:*
(i) *If $T \in \mathcal{R}_g(X)$, then $TS \in \mathcal{R}_g(X)$.*
(ii) *If $T, S \in \mathcal{R}_g(X)$, then $T + S \in \mathcal{R}_g(X)$.*

Proposition 4.1. *Let X be a non-reflexive Banach space having property* (H), *let X^* satisfy property* (H_1) *and let $T \in \mathcal{L}(X)$. Let $\omega \neq \emptyset$ be an open and connected subset of $\mathbb{C}$ such that $\sigma(T) \subset \omega$ and let $\varphi : \omega \to \mathbb{C}$ be an analytic function. If $\varphi(T)$ is a g-Riesz operator, then T is a g-g-Riesz operator.*

Proof. As φ is analytic in the compact $\sigma(T)$, it has finite number of zeros $\{\rho_1, \dots, \rho_n\}$. We can write

$$\varphi(z) = \prod_{i=1}^{n} (z - \rho_i) g(z),$$

where g is an analytic function on a neighborhood of $\sigma(T)$ such that $g(z)$ has no zero in $\sigma(T)$. Next, we define the map r by $r(z) = \frac{1}{g(z)}$, where r is an analytic function on a neighborhood of $\sigma(T)$. Therefore, we have

$$\prod_{i=1}^{n} (z - \rho_i) = \varphi(z) r(z) = r(z) \varphi(z).$$

This fact, together with [19, Lemma 6.15], proves that $\prod_{i=1}^{n} (T - \rho_i) = \varphi(T) r(T) = r(T) \varphi(T)$. Taking into account that $\varphi(T)$ is g-Riesz, it follows from Theorem 4.2 that $\prod_{i=1}^{n} (T - \rho_i)$ is g-Riesz. Accordingly to Theorem 4.1, we conclude that T is a g-g-Riesz operator with a generalized set $E = \{0, \rho_1, \dots, \rho_n\}$. □

Theorem 4.3 ([5]). *Let X be a non-reflexive Banach space having property* (H) *and let T, S be two commuting bounded linear operators on X. Then:*
(i) *If $T \in \mathcal{R}_g(X)$ and $S \in \Phi_g(X)$, then $T + S \in \Phi_g(X)$.*
(ii) *If $T \in \Phi_g(X)$ and $TS \in \mathcal{R}_g(X)$, then $S \in \mathcal{R}_g(X)$.*

Theorem 4.4. *Let X be a non-reflexive Banach space having property* (H) *and let X^* satisfy property* (H_1). *Let $T, S \in \mathcal{L}(X)$ commute and let E be a finite subset of $\mathbb{C}$ containing 0. Assume that there exists a non-zero complex polynomial P such that $P(T) \in \mathcal{R}_g(X)$. If S is a g-g-Riesz operator with a generalized set E and $P(\lambda) \neq 0$ for all $\lambda \in E$, then $S - T \in \Phi_g(X)$.*

Proof. Assume that P is a unit polynomial such that

$$P(z) = \prod_{i=1}^{n}(\lambda_i - z) \quad \text{with } \lambda_i \in \mathbb{C} \setminus E.$$

Since S is a g-g-Riesz operator, this implies that $\lambda - S \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. Furthermore, since $P(\lambda) \neq 0$ for all $\lambda \in E$, it follows that $P(S) = \prod_{i=1}^{n}(\lambda_i - S)$ with $\lambda_i \in \mathbb{C} \setminus E$. We have $\lambda_i - S \in \Phi_g(X)$ for all $i = 1, \ldots, n$. Thus, from Theorem 3.6, it follows that $P(S) \in \Phi_g(X)$. Taking into account that $P(T) \in \mathcal{R}_g(X)$ and $P(T), P(S)$ commute, using Theorem 4.3, we deduce that $P(S) - P(T) \in \Phi_g(X)$.

We can write

$$P(S) - P(T) = R(S, T)(S - T) = (S - T)R(S, T), \tag{4.2}$$

where R is a polynomial of S and T. Using equation (4.2) together with [6, Theorem 3.9], we conclude that $S - T \in \Phi_g(X)$. □

Corollary 4.1. *Let X be a non-reflexive Banach space satisfying property* (H) *and let X^* satisfy property* (H_1). *Let $T, S \in \mathcal{L}(X)$ commute and let E be a finite subset of $\mathbb{C}$ containing 0. Suppose that $T^p \in \mathcal{R}_g(X)$ for some $p \in \mathbb{N}^*$. If S is a g-g-Riesz operator with a generalized set E, then $S - T \in \Phi_g(X)$.*

Now, we establish the following theorem as a generalization of a theorem in [6] involving the class of g-g-Riesz operators.

Theorem 4.5. *Let X be a non-reflexive Banach spaces satisfying property* (H). *Assume that S is a g-g-Riesz operator and there is $\eta > 0$ such that for any $T \in \mathcal{L}(X)$ we have $\|T\| < \eta$. Then $S + T$ is g-g-Riesz operator.*

Proof. Since S is a g-g-Riesz operator, it follows that there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda - S \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. Thus, using Theorem 2.1, there exist $U \in \mathcal{L}(X)$ and $W_1, W_2 \in \mathcal{W}(X)$ such that $(\lambda - S)U = I + W_1$ and $U(\lambda - S) = I + W_2$. Hence

$$(\lambda - S - T)U = (\lambda - S)U - TU,$$
$$U(\lambda - S - T) = U(\lambda - S) - UT.$$

Taking $\eta = \|U\|^{-1}$, we get $\|UT\| < 1$ and $\|TU\| < 1$. Thus the operators $I - UT$ and $I - TU$ have bounded inverses, and we can write

$$(\lambda - S - T)U(I - TU)^{-1} = I + W_1(I - TU)^{-1},$$
$$(I - UT)^{-1}U(\lambda - S - T) = I + (I - UT)^{-1}W_2.$$

Using Lemma 2.2, we obtain $\lambda - S - T \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. Consequently, $S + T$ is a g-g-Riesz operator. □

Theorem 4.6. *Let X be a non-reflexive Banach space satisfying property* (H) *and let $T \in \mathcal{L}(X)$. If T is a g-g-Riesz operator and $W \in \mathcal{W}(X)$, then $T + W$ is a g-g-Riesz operator.*

Proof. Taking into account that T is a g-g-Riesz operator, there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda - T \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. According to Theorem 2.1, it yields that there exist $F \in \mathcal{L}(X)$ and $W_1, W_2 \in \mathcal{W}(X)$ such that $F(\lambda - T) = I + W_1$ and $(\lambda - T)F = I + W_2$. It follows that

$$F(\lambda - T - W) = F(\lambda - T) - FW = I + W_3,$$
$$(\lambda - T - W)F = (\lambda - T)F - WF = I + W_4,$$

where $W_3, W_4 \in \mathcal{W}(X)$.

Furthermore, using Lemma 2.2, we deduce that $\lambda - T - W \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. Consequently, $T + W$ is a g-g-Riesz operator. □

Theorem 4.7 ([5]). *Let X be a non-reflexive Banach space satisfying property* (H) *and let $T, S \in \mathcal{L}(X)$ be such that $ST \in \Phi_g(X)$. Then $T \in \Phi_g(X)$ if and only if $S \in \Phi_g(X)$.*

Theorem 4.8. *Let X be a non-reflexive Banach space having property* (H) *and let $P, Q \in \mathcal{L}(X)$ be such that $P^2 = P$, $Q^2 = Q$ and $T = PQ$, $S = QP$. If $T \in \Phi_g(X)$ and TS is g-g-Riesz, then T is a g-g-Riesz operator.*

Proof. As TS is g-g-Riesz, it follows that there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda - TS \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. Furthermore, using Theorem 3.6 together with the fact $T \in \Phi_g(X)$, we lead to $(\lambda - TS)T \in \Phi_g(X)$. Now, applying [20, Theorem 1.1], we find that $\lambda T - TST = \lambda T - T^2 = (\lambda - T)T \in \Phi_g(X)$. This implies by the use of Theorem 4.7 that $\lambda - T \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. Consequently, T is a g-g-Riesz. □

Now, we discuss the incidence of some perturbation results realized on the behavior of the generalized S-essential spectrum of operators in $\mathcal{C}(X)$.

Definition 4.2. Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X, Y)$ is said to have a left generalized Fredholm inverse if there exists $A \in \mathcal{L}(Y, X)$ such that $I - AT \in \mathcal{W}(X)$.

Definition 4.3. Let X be a Banach space and let $T \in \mathcal{L}(X)$. We define the set $\mathcal{I}^l_{g,T}(X)$ by

$$\mathcal{I}^l_{g,T}(X) := \{K \in \mathcal{L}(X) : K \text{ is a left generalized Fredholm inverse of } T\}.$$

Theorem 4.9. *Let X be a non-reflexive Banach space having property* (H) *and let υ be an MWNC in X. Let $T, S \in \mathcal{L}(X)$. If S is a g-g-Riesz operator, let $L \in \mathcal{I}^l_{g,\lambda-S}(X)$ and let $\varpi(TL) < 1$. Then $T + S$ is g-g-Riesz operator.*

Proof. As S is a g-g-Riesz operator, there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda - S \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. Taking into account that $\varpi(TL) < 1$, it follows from Theorem 2.2 that $(I - TL) \in \Phi_g(X)$. By using Theorem 3.6, we get $(I - TL)(\lambda - S) \in \Phi_g(X)$. Since $L \in \mathcal{I}^l_{g,\lambda-S}(X)$, there exists $W \in \mathcal{W}(X)$ such that $L(\lambda - S) = I - W$. On the other hand, we have

$$(I - TL)(\lambda - S) = \lambda - S - TL(\lambda - S) = \lambda - S - T + TW.$$

Then we get $\lambda - S - T + TW \in \Phi_g(X)$. Hence, using Theorem 3.6, we obtain $\lambda - T - S \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. Consequently, $T + S$ is a g-g-Riesz operator. □

Theorem 4.10. *Let X be a non-reflexive Banach space having property* (H). *Let $T, S \in \mathcal{L}(X)$ and let E be a finite subset of $\mathbb{C}$ containing 0. If for every $\lambda \in \Phi_{g,T}(X) \setminus E$ there exists $U \in \mathcal{I}^l_{g,\lambda-T}(X)$ and SU is a g-g-Riesz operator with a generalized set E, then*

$$\sigma_{e_4,g}(T + S) \setminus E \subseteq \sigma_{e_4,g}(T) \setminus E.$$

Proof. Taking into account that SU is g-g-Riesz with a generalized set E, we have $(\lambda - SU) \in \Phi_g(X)$ for all $\lambda \in \mathbb{C} \setminus E$. Since $U \in \mathcal{I}^l_{g,\lambda-T}(X)$, there exists $W_1 \in \mathcal{W}(X)$ such that $U(\lambda - T) = I - W_1$. Furthermore, we can write

$$\lambda - T - S = (I - SU)(\lambda - T) - SW_1. \tag{4.3}$$

In fact, let $\lambda \notin \sigma_{e_4,g}(T) \setminus E$. Then $\lambda - T \in \Phi_g(X)$. Using Theorem 3.6, we obtain $(\lambda - SU)(\lambda - T) \in \Phi_g(X)$. Due to $SW_1 \in \mathcal{W}(X)$, we have $\lambda - T - S \in \Phi_g(X)$. This means $\lambda - T - S \in \Phi_g(X)$. Consequently, $\lambda \notin \sigma_{e_4,g}(T + S) \setminus E$. □

Remark 4.2 ([5]). Let X be a Banach space and $T \in \mathcal{L}(X)$. Then $T \in \mathcal{R}_g(X)$ if and only if $I + \lambda T \in \Phi_g(X)$ for each λ.

Now, we give a relation between the generalized Wolf essential spectrum and the left (resp. the right) essential spectrum of an operator acting on a Banach space X by means of g-Riesz operators. In what follows, for $T \in \mathcal{L}(X)$, we define the following sets:

$$\sigma^*_{e_4}(T) := \bigcap_{W \in \mathcal{W}(X)} \sigma(T + W),$$

$$\Omega^L_T(X) := \{W \in \mathcal{L}(X) : -(\lambda - T - W)^{-1} W \in \mathcal{R}_g(X) \text{ for all } \lambda \in \varrho(T + W)\},$$

$$\Omega^R_T(X) := \{W \in \mathcal{L}(X) : -W(\lambda - T - W)^{-1} \in \mathcal{R}_g(X) \text{ for all } \lambda \in \varrho(T + W)\}.$$

Moreover, we denote

$$\sigma_L(T) := \bigcap_{W\in\Omega_T^L(X)} \sigma(T+W) \quad \text{and} \quad \sigma_R(T) := \bigcap_{W\in\Omega_T^R(X)} \sigma(T+W).$$

Theorem 4.11. *Let X be a non-reflexive Banach space satisfying property* (H) *and let $T\in\mathcal{L}(X)$. Then:*
(i) $\sigma_{e_4,g}(T)\subset\sigma_L(T)\subset\sigma_{e_4}^*(T)$,
(ii) $\sigma_{e_4,g}(T)\subset\sigma_R(T)\subset\sigma_{e_4}^*(T)$.

Proof. (i) For $\lambda\in\varrho(T+W)$, we have

$$\lambda - T = (\lambda - T - W)(I + (\lambda - T - W)^{-1}W). \tag{4.4}$$

Obviously, $\mathcal{W}(X)\subset\Omega_T^L(X)$, which implies that

$$\bigcap_{W\in\Omega_T^L(X)} \sigma(T+W)\subset\sigma_{e_4}^*(T).$$

Now, we prove that

$$\sigma_{e_4,g}(T)\subset\bigcap_{W\in\Omega_T^L(X)} \sigma(T+W).$$

Let $\lambda\notin\bigcap_{W\in\Omega_T^L(X)}\sigma(T+W)$. Then there exists $W\in\Omega_T^L(X)$ such that $\lambda\notin\sigma(T+W)$. So, $\lambda - T - W\in\Phi_g(X)$. Furthermore, it follows from the fact $-(\lambda - T - W)^{-1}W\in\mathcal{R}_g(X)$ together with Remark 4.2 that

$$I + (\lambda - T - W)^{-1}W\in\Phi_g(X).$$

Now, the use of equation (4.4) together with Theorem 3.6 enables us to conclude that $\lambda - T\in\Phi_g(X)$, which allows us to get $\lambda\notin\sigma_{e_4,g}(T)$. Consequently, $\sigma_{e_4,g}(T)\subset\bigcap_{W\in\Omega_T^L(X)}\sigma(T+W)$.

(ii) Proceeding like in the proof of (i), we obtain the desired result by replacing equation (4.4) by

$$(\lambda - T) = (I + W(\lambda - T - W)^{-1})(\lambda - T - W),$$

which completes the proof. □

References

[1] B. Abdelmoumen, A. Jeribi and M. Mnif, On graph measures in Banach spaces and description of essential spectra of multidimensional transport equation, *Acta Math. Sci. Ser. B (Engl. Ed.)* **32** (2012), no. 5, 2050–2064.
[2] P. Aiena, A characterization of Riesz operators, *Math. Z.* **196** (1987), no. 4, 491–496.
[3] P. Aiena and U. Oliveri, Generalized Riesz operators on Banach spaces, *Rend. Semin. Mat. Univ. Padova* **61** (1979), 303–312.
[4] K. Astala, On measures of noncompactness and ideal variations in Banach spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math. Diss.* **29** (1980), 1–42.
[5] A. Azzouz, M. Beghdadi and B. Krichen, g-Riesz operators and their spectral properties, *Adv. Oper. Theory* **7** (2022), no. 2, Paper No. 18.
[6] A. Azzouz, M. Beghdadi and B. Krichen, Generalized relative essential spectra, *Filomat* **36** (2022), no. 8, 2657–2673.
[7] J. Banaś, A. B. Ali, K. Mahfoudhi and B. Saadaoui, (P, Q) - ε-pseudo condition spectrum for 2×2 matrices. Linear operator and application, *J. Contemp. Math. Anal.* **58** (2023), no. 4, 217–225.
[8] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes Pure Appl. Math. 60, Marcel Dekker, New York, 1980.
[9] J. Banaś and J. Rivero, On measures of weak noncompactness, *Ann. Mat. Pura Appl. (4)* **151** (1988), 213–224.
[10] S. R. Caradus, Operators of Riesz type, *Pacific J. Math.* **18** (1966), 61–71.
[11] P. Constantin, A. Debussche, G. P. Galdi, M. Růžička and G. Seregin, *Topics in Mathematical Fluid Mechanics*, Lecture Notes in Math. 2073, Springer, Heidelberg, 2013.
[12] F. S. De Blasi, On a property of the unit sphere in a Banach space, *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.)* **21(69)** (1977), no. 3–4, 259–262.
[13] G. Emmanuele, Measure of weak noncompactness and fixed point theorems, *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.)* **25(73)** (1981), no. 4, 353–358.

[14] M. González and A. Martínez-Abejón, *Tauberian Operators*, Oper. Theory Adv. Appl. 194, Birkhäuser, Basel, 2010.
[15] A. Jeribi, A characterization of the Schechter essential spectrum on Banach spaces and applications, *J. Math. Anal. Appl.* **271** (2002), no. 2, 343–358.
[16] D. Kleinecke, Almost-finite, compact, and inessential operators, *Proc. Amer. Math. Soc.* **14** (1963), 863–868.
[17] A. F. Ruston, Operators with a Fredholm theory, *J. Lond. Math. Soc.* **29** (1954), 318–326.
[18] M. Schechter, Riesz operators and Fredholm perturbations, *Bull. Amer. Math. Soc.* **74** (1968), 1139–1144.
[19] M. Schechter, *Principles of Functional Analysis*, Academic Press, New York, 1971.
[20] C. Schmoeger, Common spectral properties of linear operators A and B such that $ABA = A^2$ and $BAB = B^2$, *Publ. Inst. Math. (Beograd) (N. S.)* **79(93)** (2006), 109–114.
[21] K. W. Yang, The generalized Fredholm operators, *Trans. Amer. Math. Soc.* **216** (1976), 313–326.