Research Article

Rabeb Aydi, Imen Ferjani and Bilel Krichen*

Index for generalized Fredholm operators and generalized perturbation theory

https://doi.org/10.1515/gmj-2023-2021
Received September 27, 2022; revised December 6, 2022; accepted December 12, 2022

Abstract: In this paper, we develop some generalized Fredholm perturbation results for bounded linear operators acting on non-reflexive Banach spaces satisfying certain conditions. Furthermore, we investigate the stability of the generalized Schechter S-essential spectrum by means of the so-called generalized Fredholm index. This concept is introduced as a semigroup homomorphism satisfying certain properties. Some generalized Fredholm results for block operator matrices acting on a non-reflexive product of two Banach spaces are also developed.

Keywords: Generalized Fredholm operator, index, generalized Fredholm perturbation, generalized essential spectrum, matrix operator

MSC 2020: 47A53, 47A10, 47A55

1 Introduction

The perturbation theory has developed into an important field of study in both pure and applied mathematics. It has been revealed as a very powerful and important tool in the study of the behavior of spectral properties of linear operators undergoing a small change. Various aspects of perturbations for linear operators may be found in [11, 14, 17]. In particular, the Fredholm perturbation classes, introduced in [10], played an important role in the description of the essential spectrum of bounded or unbounded linear operators. In 1976, Yang [24] introduced and studied the concept of generalized Fredholm operators for bounded linear operators acting on Banach spaces as an extension of the notion of Fredholm operators. This definition asserts that a bounded linear operator is a generalized Fredholm operator if its range is closed and both its kernel and co-kernel are reflexive. Based on this definition, we are mainly interested in the study of a large class of Fredholm operators by introducing the class of generalized Fredholm perturbations for bounded linear operators acting on non-reflexive Banach spaces satisfying certain topological properties.

First, let us recall some standard notations from the Fredholm and generalized Fredholm theories needed in this paper (see, for instance, [1, 17, 22, 24]). Let X and Y be two Banach spaces. The set of all bounded (resp. compact), linear operators from X into Y is denoted by $\mathcal{L}(X, Y)$ (resp. $\mathcal{K}(X, Y)$). The dual (resp. the bidual) is denoted by X^* (resp. X^{**}), T^* is the conjugate of an operator T, and T^{**} is the second conjugate. For $T \in \mathcal{L}(X, Y)$, we use $\alpha(T)$ and $\beta(T)$ to denote the dimension of the kernel $\mathcal{N}(T)$ and the codimension of the range $\mathcal{R}(T)$ in Y, respectively. An operator $T \in \mathcal{L}(X, Y)$ is called a Fredholm operator if it has a finite-dimensional kernel space and a closed range with finite codimension; its index is given by $i(T) = \alpha(T) - \beta(T)$. The operator T is said to be weakly compact if $T(M)$ is relatively weakly compact in Y for every bounded subset M of X. The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X, Y)$. If $X = Y$, this family of operators, denoted by $\mathcal{W}(X) := \mathcal{W}(X, X)$, is a closed two-sided ideal of $\mathcal{L}(X)$ containing the closed ideal of compact operators (see [8]).

***Corresponding author: Bilel Krichen,** Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Road Soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia, e-mail: bilel.krichen@fss.usf.tn
Rabeb Aydi, Imen Ferjani, Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Road Soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia, e-mail: rabeb.aydi.2019@gmail.com, imenoferjani@yahoo.fr

Further, an operator T is said to be Tauberian whenever T^{**} preserves the naturel embedding of X into its double dual, i.e., $x \in X^*$, $T^{**}x \in Y$ implies $X \in X$. It is immediate that if T is a Tauberian operator and $T^{**}x = 0$, $x \in X^{**}$, then $x \in X$. This implies that $\mathcal{N}(\mathcal{T})$ is reflexive. In particular, if $\mathcal{R}(T)$ is closed, then T is Tauberian if and only if $\mathcal{N}(T)$ is reflexive. Also, T is co-Tauberian if its conjugate T^* is Tauberian. Furthermore, T is said invertible modulo weakly compact operators if there exist $W_1 \in \mathcal{W}(X)$, $W_2 \in \mathcal{W}(Y)$ and an operator $T_0 \in \mathcal{L}(Y, X)$ such that $T_0T = I + W_1$ and $TT_0 = I + W_2$. If S is a non-null bounded linear operator from X into Y and $T \in \mathcal{L}(X, Y)$, we define the S-resolvent set of T by

$$\rho_S(T) := \{\lambda \in \mathbb{C} : \lambda S - T \text{ has a bounded inverse}\},$$

and the S-spectrum of T by

$$\sigma_S(T) := \mathbb{C} \setminus \rho_S(T).$$

Next, using the notations given in [5], we define the class of upper generalized semi-Fredholm operators from X into Y by

$$\Phi_{g+}(X, Y) := \{T \in \mathcal{L}(X, Y) : \mathcal{N}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\},$$

and the class of lower generalized semi-Fredholm operators from X into Y by

$$\Phi_{g-}(X, Y) := \{T \in \mathcal{L}(X, Y) : Y/\mathcal{R}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\}.$$

Here, we denote by $\Phi_g(X, Y) := \Phi_{g+}(X, Y) \cap \Phi_{g-}(X, Y)$ the set of generalized Fredholm operators in $\mathcal{L}(X, Y)$, and by $\Phi_{g\pm}(X, Y) := \Phi_{g+}(X, Y) \cup \Phi_{g-}(X, Y)$ the set of generalized semi-Fredholm operators. If $X = Y$, the sets $\mathcal{L}(X, Y)$, $\mathcal{K}(X, Y)$, $\Phi_g(X, X)$, $\Phi_{g+}(X, Y)$, $\Phi_{g-}(X, Y)$, and $\Phi_{g\pm}(X, Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\Phi_g(X)$, $\Phi_{g+}(X)$, $\Phi_{g-}(X)$, and $\Phi_{g\pm}(X)$, respectively. Observe that, when X is a reflexive space, $\Phi_g(X)$ simply consists of linear operators having closed ranges. Furthermore, we note that in an infinite-dimensional Banach space X we have $\Phi(X) \subsetneq \Phi_g(X)$, where $\Phi(X)$ is the set of Fredholm operators on X. Indeed, for $1 < p < \infty$, there exists a projection $P \in \mathcal{L}(L_p)$ into the closed subspace generated by Rademacher functions (see the proof of [2, Proposition 6.4.2]). The operator $I - P : L_p \to L_p$ has a closed range since $\mathcal{R}(I - P) = \mathcal{N}(P)$. Hence, by the reflexivity of the space L_p, we deduce that $I - P \in \Phi_g(L_p)$. On the other hand, $\mathcal{N}(I - P)$ is the closed subspace generated by the Rademacher functions which is strongly embedded (see [2, Proposition 6.4.5]). Hence, $I - P$ has an infinite-dimensional kernel, and therefore $I - P$ is not Fredholm.

Now, for $S \in \mathcal{L}(X, Y)$, $S \neq 0$, a complex number λ is in $\Phi_{g+,T,S}$, $\Phi_{g-,T,S}$, $\Phi_{g\pm,T,S}$, or $\Phi_{g,T,S}$, if $\lambda S - T$ is in $\Phi_{g+}(X, Y)$, $\Phi_{g-}(X, Y)$, $\Phi_{g\pm}(X, Y)$, or $\Phi_g(X, Y)$, respectively. If $S = I$, then the sets $\Phi_{g+,T,S}$, $\Phi_{g-,T,S}$, $\Phi_{g\pm,T,S}$, and $\Phi_{g,T,S}$ are simply denoted by $\Phi_{g+,T}$, $\Phi_{g-,T}$, $\Phi_{g\pm,T}$, and $\Phi_{g,T}$, respectively. We say that X has the property (H_1) (resp. (H_2)) if every reflexive subspace admits a closed complementary subspace (resp. if every closed subspace with reflexive quotient space admits a closed complementary subspace). We say that X has the property (H) if it satisfies both properties (H_1) and (H_2). For example, the $L_p(0, 1)$ spaces for $1 < p < \infty$ have the property (H) (see [21]).

In [24], Yang characterized the set of generalized Fredholm operators from X into Y as those bounded operators invertible modulo the two-sided ideal of weakly compact operators having closed range. More precisely, it was proved in [24, Theorem 5.10] that if X and Y are two Banach spaces satisfying the properties (H_1) and (H_2), respectively, and $T \in \mathcal{L}(X, Y)$, then $T \in \Phi_g(X, Y)$ if and only if T is invertible modulo weakly compact operators and $\mathcal{R}(T)$ is closed. Also, in [24], Yang proved that the set of generalized Fredholm operators is also an intersection of the co-Tauberian operators with closed range and Tauberian operators. Recently, Azzouz, Beghdadi and Krichen [5] continued this study to investigate the generalized S-essential spectra of a bounded linear operator $T \in \mathcal{L}(X, Y)$ with respect to a given linear operator $S \in \mathcal{L}(X, Y)$. In fact, their study was restricted to the generalized Wolf S-essential spectrum and the generalized Gustafson S-essential spectrum, which are respectively defined by

$$\sigma_{Se_4,g}(T) := \{\lambda \in \mathbb{C} : (\lambda S - T) \notin \Phi_g(X)\} := \mathbb{C} \setminus \Phi_{g,T,S},$$
$$\sigma_{Se_1,g}(T) := \{\lambda \in \mathbb{C} : (\lambda S - T) \notin \Phi_{g+}(X)\} := \mathbb{C} \setminus \Phi_{g+,T,S}.$$

Note that if $S = I$, the two previous generalized essential spectra of T will be respectively replaced by $\sigma_{e_4,g}(T)$ and $\sigma_{e_1,g}(T)$.

The first aim of this work is to investigate the stability of the generalized Schechter S-essential spectrum by defining the so-called generalized Fredholm index. This axiomatic definition may be seen as an extension of the classical Fredholm index. It is introduced as a semigroup homomorphism satisfying certain properties and is defined on the set of generalized Fredholm operators acting on a non-reflexive Banach space satisfying the property (H). Moreover, inspired by the method developed in [13], we show that the generalized Wolf essential spectrum satisfies the spectral mapping theorem for complex-valued functions which are locally holomorphic on an open set containing the spectrum.

The second aim of this paper is to apply our results to investigate the generalized essential spectrum of 2×2 block operator matrices acting on a non-reflexive product of two Banach spaces. Our results extend many known ones in the literature, in particular, those obtained in the works [1, 16, 19, 20] and the books [14, 23].

This paper is organized in the following way. In Section 2, we recall some definitions and results needed throughout this paper. In Section 3, we establish some generalized Fredholm perturbation results on non-reflexive Banach spaces. We apply these results to discuss the stability of the generalized Schechter essential spectrum. Further, we investigate the spectral mapping theorem for the generalized Wolf essential spectrum and the generalized Schechter essential spectrum. In Section 4, we provide some sufficient conditions for a 2×2 block operator matrices to be generalized Fredholm.

2 Preliminaries

The concept of measure of weak noncompactness has many applications in operator theory, fixed point theory and the theories of differential and integral equations (see, e.g., [3, 6, 9, 15]). It was originally introduced by De Blasi [7]. Now, let us recall the axiomatic approach in defining measures of weak noncompactness. First, let us denote by X a Banach space and by X^* its topological dual. Let $B_r = B(0, r)$ be the open ball centered at 0 with radius r and let $\overline{B}_r$ be its closure. Let the family of all nonempty bounded subsets of X be denoted by $\mathcal{M}_X$ and its subfamily of all relatively weakly compact sets by $\mathcal{M}_X^w$. Moreover, let $\operatorname{conv}(A)$ denote the convex hull of a subset A of X.

Definition 2.1 ([6]). Let X be a Banach space and let $\mu : \mathcal{M}_X \to [0, +\infty[$ be a real-valued function. We call μ a measure of weak noncompactness on X if the following conditions are satisfied for any elements A, B of $\mathcal{M}_X$:
(i) $\mu(A) = 0$ if and only if A is relatively weakly compact.
(ii) If $A \subset B$, then $\mu(A) \leq \mu(B)$.
(iii) $\mu(\overline{\operatorname{conv}(A)}) = \mu(A)$.
(iv) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.
(v) $\mu(A + B) \leq \mu(A) + \mu(B)$.
(vi) $\mu(\lambda A) = |\lambda|\mu(A)$ for $\lambda \in \mathbb{C}$.

As an example of a measure of weak noncompactness in a Banach space X, we have the measure of noncompactness defined by De Blasi (see [7]) in the following formula:

$$\omega(A) = \inf\{r > 0 : \text{there exists } N \in \mathcal{M}_X^w \text{ such that } A \subset N + \overline{B}_r\}.$$

Moreover, we can define the measure of weak noncompactness of a bounded linear operator T, denoted by $\overline{\omega}(T)$, as follows:

$$\overline{\omega}(T) = \inf\{k : \omega(T(A)) \leq k\omega(A) \text{ for all } A \in \mathcal{M}_X\}.$$

In the next proposition, we recall some properties of $\overline{\omega}(\cdot)$.

Proposition 2.2 ([7]). *Let X be a Banach space, let $T, S \in \mathcal{L}(X)$ and let $B \in \mathcal{M}_X$. Then we have the following properties:*
(i) $\overline{\omega}(T) = 0$ *if and only if T is weakly compact.*
(ii) $\omega(T(B)) \leq \overline{\omega}(T)\omega(B)$.
(iii) $\overline{\omega}(TS) \leq \overline{\omega}(T)\overline{\omega}(S)$.

(iv) $\overline{\omega}(T+S) \leq \overline{\omega}(T) + \overline{\omega}(S)$.
(v) $\overline{\omega}(\lambda T) = |\lambda|\overline{\omega}(T)$, *for* $\lambda \in \mathbb{C}$.

Definition 2.3 ([4]). Let X be a non-reflexive Banach space and let $T \in \mathcal{L}(X)$. We introduce the following non-negative quantities:

$$\overline{\beta}(T) := \inf\Big\{\frac{\omega(T(A))}{\omega(A)} : A \in \mathcal{M}_X \text{ and } \omega(A) > 0\Big\}.$$

Theorem 2.4 ([5]). *Let X be a non-reflexive Banach space and let $T \in \mathcal{L}(X)$. The following statements hold:*
(i) *If $\overline{\beta}(T) > 0$, then $T \in \Phi_{g+}(X)$.*
(ii) *If X satisfies the property* (H_1) *and $T \in \Phi_{g+}(X)$. then $\overline{\beta}(T) > 0$.*

From this theorem, we immediately obtain the following properties.

Theorem 2.5 ([5]). *Let X be a non-reflexive Banach and let $T, S \in \mathcal{L}(X)$.*
(i) *If $ST \in \Phi_{g+}(X)$ and X satisfies the property* (H_1)*, then $T \in \Phi_{g+}(X)$.*
(ii) *If $ST \in \Phi_{g-}(X)$ and X^* satisfies the property* (H_1)*, then $S \in \Phi_{g-}(X)$.*
(iii) *If $ST \in \Phi_{g}(X)$ and X and X^* satisfy the property* (H_1)*, then $S \in \Phi_{g-}(X)$ and $T \in \Phi_{g+}(X)$.*

Theorem 2.6 ([5]). *Let X, Y and Z be three non-reflexive Banach spaces and let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$. Then the following statements hold:*
(i) *Suppose X, Y and Z satisfy the properties* (H_1), (H) *and* (H_2)*, respectively. If $T \in \Phi_g(X, Y)$ and $S \in \Phi_g(Y, Z)$, then $ST \in \Phi_g(X, Z)$.*
(ii) *Suppose X and Y satisfy the properties* (H_1) *and* (H_2)*, respectively. If $T \in \Phi_g(X, Y)$ and $W \in \mathcal{W}(X, Y)$, then $T + W \in \Phi_g(X, Y)$.*

For $X = Y = Y$, we have the following assertions:
(i) *Suppose X satisfies the property* (H_1)*. If $T \in \Phi_{g+}(X)$ and $S \in \Phi_{g+}(X)$, then $ST \in \Phi_{g+}(X)$.*
(ii) *Suppose X satisfies the property* (H)*. If $T \in \Phi_{g-}(X)$ and $S \in \Phi_{g-}(X)$, then $ST \in \Phi_{g-}(X)$.*

The next proposition gives another equivalent definition to the generalized Gustafson S-essential spectrum.

Proposition 2.7 ([5]). *Let X be a non-reflexive Banach space having the property* (H_1) *and let T and S be two bounded linear operators on X. Then*

$$\sigma_{Se_1,g}(T) = \bigcap_{W \in \mathcal{W}(X)} \sigma_{Se_4,g}(T + W).$$

To end this section, we recall the following result established in [5].

Proposition 2.8. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ such that $\overline{\omega}(T^n) < 1$ for some $n \in \mathbb{N} \setminus \{0\}$. Then $(I - T) \in \Phi_g(X)$.*

3 Some generalized Fredholm perturbation results

We begin with the following lemma.

Lemma 3.1. *Let X be a Banach space. If X satisfies the property* (H_2)*, then X^* satisfies the property* (H_1)*.*

Proof. Suppose that X satisfies the property (H_2). For a subspace M of X and a subspace N of X^*, we call

$$M^\circ = \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in M\} \subset X^*$$

the annihilator of M, and we call

$${}^\circ N = \{x \in X : x^*(x) = 0 \text{ for all } x^* \in N\} \subset X$$

the pre-annihilator of N. Let N be a reflexive subspace of X^*. Obviously, ${}^\circ N$ is closed in X. Then $(X/{}^\circ N)^*$ is isomorphic to $({}^\circ N)^\circ$. By the reflexivity of N, we have $({}^\circ N)^\circ = N$ (see [22, Theorem 8.5]). Consequently, [22, Theorem 8.1] yields that $X/{}^{\circ}N$ is reflexive. Now, since X satisfies the property (H_2), there exists a closed subspace M of X such that

$$X = {}^{\circ}N \oplus M.$$

Therefore, from [11, Lemma 4.1.11], we obtain

$$X^* = ({}^{\circ}N)^{\circ} \oplus M^{\circ} = N \oplus M^{\circ}.$$

Thus, N is complemented in X^*. □

Remark 3.2. When X is a reflexive space, it is easy to check that if X satisfies the property (H_1), then X^* does so too.

Definition 3.3 ([13]). Let $\mathcal{B}$ be a Banach algebra. An open semigroup $\mathcal{U}$ of $\mathcal{B}$ will be called an F-semigroup if the following conditions hold:
(i) $a, b \in \mathcal{B}$ and $ab = ba \in \mathcal{U}$ imply $b \in \mathcal{U}$ and $b \in \mathcal{U}$.
(ii) There exists a closed ideal $\mathcal{R}$ such that $\mathcal{U} = q^{-1}(S)$, where S is an open semigroup in $\mathcal{B}/\mathcal{R}$ containing G (here G is the group of invertible elements in $\mathcal{B}/\mathcal{R}$) and having the property that $S \setminus G$ is also open, and q is the canonical homomorphism of $\mathcal{B}$ into $\mathcal{B}/\mathcal{R}$.

We now prove that $\Phi_g(X)$ is an F-semigroup in $\mathcal{L}(X)$, where X is a non-reflexive Banach space satisfying the property (H).

Example 3.4. Let X be a non-reflexive Banach space satisfying the property (H). Then $\Phi_g(X)$ is an F-semigroup in $\mathcal{L}(X)$. Indeed, it follows from [5, Theorem 3.12] that $\Phi_g(X)$ is an open semigroup. Next, let $T, S \in \mathcal{L}(X)$ such that $ST = TS \in \Phi_g(X)$. Since X satisfies the property (H), Lemma 3.1 yields that X^* satisfies the property (H_1). Hence, we can apply Theorem 2.5 (iii) and deduce that $T \in \Phi_g(X)$ and $S \in \Phi_g(X)$. Now, since X satisfies the property (H), it follows from [24, Theorem 5.10] that $\Phi_g(X) = \pi^{-1}(G)$, where G is the group of invertible elements in $\mathcal{L}(X)/\mathcal{W}(X)$ and π is the canonical homomorphism of $\mathcal{L}(X)$ into $\mathcal{L}(X)/\mathcal{W}(X)$. Therefore, $\Phi_g(X)$ satisfies assumptions (i) and (ii) of Definition 3.3, and hence is an F-semigroup.

Now we will introduce an abstract index for generalized Fredholm operators.

Definition 3.5. Let X be a non-reflexive Banach space satisfying the property (H). We define the generalized Fredholm index on the semigroup $\Phi_g(X)$ by the semigroup homomorphism $i_g : \Phi_g(X) \to \mathbb{Z}$ such that the following conditions hold:
(i) $i_g(T) = 0$ for all invertible elements T in $\mathcal{L}(X)$.
(ii) $i_g(I + W) = 0$ for all W in $\mathcal{W}(X)$.

Lemma 3.6. *Let X be a non-reflexive Banach space having the property* (H_1) *and let $T \in \mathcal{L}(X)$. Suppose that $T \in \Phi_g(X)$ and let $T_0 \in \mathcal{L}(X)$ be any operator satisfying*

$$TT_0 = I + W_1, \tag{3.1}$$
$$T_0T = I + W_2, \tag{3.2}$$

where $W_1, W_2 \in \mathcal{W}(X)$. Then $T_0 \in \Phi_g(X)$ and $i_g(T_0) = -i_g(T)$.

Proof. By hypothesis, equations (3.1) and (3.2) hold with $W_1, W_2 \in \mathcal{W}(X)$. Then, by equation (3.1), we get $TT_0 \in \Phi_g(X)$. Since X satisfies the property (H_1), it follows from Theorem 2.5 (i) that $T_0 \in \Phi_{g+}(X)$. So, by applying [24, Theorem 5.10], we deduce that $T_0 \in \Phi_g(X)$. Moreover,

$$i_g(T_0) + i_g(T) = i_g(I + W_1) = 0.$$

Hence, $i_g(T_0) = -i_g(T)$. □

Remark 3.7. Let X be a non-reflexive Banach space having the property (H). From the above definition it is easy to deduce that

$$i_g(T + W) = i_g(T) \quad \text{for each } T \in \Phi_g(X) \text{ and } W \in \mathcal{W}(X).$$

Indeed, let $T \in \Phi_g(X)$. Since X satisfies the property (H), from [24, Theorem 5.10] there exist $S \in \mathcal{L}(X)$ and $W_1 \in \mathcal{W}(X)$ such that $ST = I + W_1$. This leads to $S(T + W) = I + W_1 + SW$. On the other hand, by Lemma 3.6, we have $S \in \Phi_g(X)$ and $i_g(S) = -i_g(T)$. Hence, by using Theorem 2.6 (ii), we infer that

$$T + W \in \Phi_g(X) \quad \text{and} \quad i_g(S) + i_g(T + W) = i_g(I + W_1 + SW) = i_g(I) = 0.$$

Therefore, $i_g(I + W) = i_g(T)$.

Next, we will prove a theorem describing the properties of the generalized Fredholm index.

Theorem 3.8. *Let X be a non-reflexive Banach space having the property* (H) *and let $S \in \mathcal{L}(X)$. Suppose that $S \in \Phi_g(X)$.*

(i) *There is $\eta > 0$ such that for any $T \in \mathcal{L}(X)$ satisfying $\|T\| < \eta$, one has $(S + T) \in \Phi_g(X)$ and*

$$i_g(S + T) = i_g(S). \tag{3.3}$$

(ii) *The map $i_g : \Phi_g(X) \to \mathbb{Z}$ is continuous.*

Proof. (i) Suppose $S \in \Phi_g(X)$. Then there exist $S_0 \in \mathcal{L}(X)$ and $W_1, W_2 \in \mathcal{W}(X)$ such that $S_0 S = I + W_1$ and $SS_0 = I + W_2$. Azzouz et al. proved in [5, Theorem 3.12] that the operators $I + S_0 T$ and $I + TS_0$ have bounded inverses, and thus

$$(I + S_0 T)^{-1} S_0 (S + T) = I + (I + S_0 T)^{-1} W_1,$$
$$(S + T) S_0 (I + TS_0)^{-1} = I + W_2 (I + TS_0)^{-1}.$$

Therefore, $(S + T) \in \Phi_g(X)$ and

$$i_g((I + S_0 T)^{-1}) + i_g(S_0) + i_g(S + T) = 0.$$

Furthermore, by using Lemma 3.6, we have $i_g(S_0) = -i_g(S)$. Since $i_g((I + S_0 T)^{-1}) = 0$, we see that equation (3.3) holds.

(ii) Let $(T_n)_{n\in\mathbb{N}}$ be any sequence included in $\Phi_g(X)$ such that $T_n \to T$ as $n \to \infty$. So, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|T_n - T\| < \varepsilon.$$

Now, since $\Phi_g(X)$ is an open set, there is $\eta > 0$ such that

$$B(T_n, \eta) \subset \Phi_g(X).$$

For $\varepsilon = \eta$, we have

$$\|T_n - T\| < \eta \quad \text{for all } n \geq N.$$

Hence, it follows from assertion (i) that

$$i_g(T_n) = i_g(T_n - T + T) = i_g(T) \quad \text{for all } n \geq N.$$

Consequently, i_g is sequentially continuous. □

Remark 3.9. Theorem 3.8 is an extension of [22, Theorem 5.11] for generalized Fredholm operators.

Proposition 3.10. *Let X be a non-reflexive Banach space having the property* (H) *and let T, S be two bounded linear operators on X, where S is a nonzero operator. Then $i_g(\lambda S - T)$ is constant on any component of $\Phi_{g,T,S}$.*

Proof. Let λ_1, λ_2 be any points in $\Phi_{g,T,S}$ which are connected by a smooth curve C whose points are all in $\Phi_{g,T,S}$. Then, for each $\lambda \in C$, there is $\varepsilon > 0$ such that $\mu \in \Phi_{g,T,S}$ and $i_g(\mu S - T) = i_g(\lambda S - T)$ for all μ satisfying $|\mu - \lambda| < \frac{\varepsilon}{\|S\|}$. By the Heine–Borel theorem, there exists a finite number of such sets which cover C. Since each of these sets overlaps with, at least, one other set and since $i_g(\mu S - T)$ is constant on each of them, we see that $i_g(\lambda_1 S - T) = i_g(\lambda_2 S - T)$. □

The rest of this section is devoted to some generalized Fredholm perturbation results and the stability of the generalized S-essential spectra of bounded linear operators acting on a non-reflexive Banach space.

Definition 3.11. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$. Then F is called a generalized Fredholm perturbation if $U + F \in \Phi_g(X, Y)$ whenever $U \in \Phi_g(X, Y)$. Moreover, F is called an upper (resp. lower) generalized Fredholm perturbation if $U + F \in \Phi_{g+}(X, Y)$ (resp. $U + F \in \Phi_{g-}(X, Y)$) whenever $U \in \Phi_{g+}(X, Y)$ (resp. $U \in \Phi_{g-}(X, Y)$).

The sets of generalized Fredholm, generalized upper Fredholm and generalized lower Fredholm perturbations are denoted by $\mathcal{F}_g(X, Y)$, $\mathcal{F}_{g+}(X, Y)$ and $\mathcal{F}_{g-}(X, Y)$, respectively. If $X = Y$, then $\mathcal{F}_g(X, Y)$, $\mathcal{F}_{g+}(X, Y)$ and $\mathcal{F}_{g-}(X, Y)$ are replaced by $\mathcal{F}_g(X)$, $\mathcal{F}_{g+}(X)$ and $\mathcal{F}_{g-}(X)$, respectively.

Remark 3.12. Let X be a non-reflexive Banach space.

(i) If X satisfies the property (H), then
$$\mathcal{F}(X) \subset \mathcal{F}_g(X),$$
where $\mathcal{F}(X)$ is the set of Fredholm perturbations on X. Indeed, let $F \in \mathcal{F}(X)$ and $T \in \Phi_g(X)$. Then there exists $T_0 \in \mathcal{L}(X)$ satisfying equations (3.1) and (3.2) with $W_1, W_2 \in \mathcal{W}(X)$. From equation (3.1), we have $(T + F)T_0 = I + W_1 + FT_0$. Since $\mathcal{F}(X)$ is an ideal in $\mathcal{L}(X)$ (see [4]), we deduce that $I + FT_0 \in \Phi(X)$, and hence $I + FT_0 \in \Phi_g(X)$. By using Theorem 2.5, we infer that $T + F \in \Phi_{g-}(X)$. Similarly, we prove that
$$T_0(T + F) \in \Phi_g(X),$$
and hence $T + F \in \Phi_{g+}(X)$. Therefore, $T + F \in \Phi_g(X)$.

(ii) If X satisfies the property (H_1) (resp. X^* satisfies the property (H_1)), then we have $\mathcal{W}(X) \subset \mathcal{F}_{g+}(X) \subset \mathcal{F}_g(X)$ (resp. $\mathcal{W}(X) \subset \mathcal{F}_{g-}(X) \subset \mathcal{F}_g(X)$).

Theorem 3.13. *Let X be a non-reflexive Banach space satisfying the property* (H) *and let $T$and S be two bounded linear operators on X such that $ST \in \Phi_g(X)$. Then $T \in \Phi_g(X)$ if and only if $S \in \Phi_g(X)$.*

Proof. Suppose that $T \in \Phi_g(X)$. From the fact that X satisfies the property (H), we infer that there exists $T_0 \in \mathcal{L}(X)$ satisfying equations (3.1) and (3.2) with $W_1, W_2 \in \mathcal{W}(X)$. From equation (3.2), we write
$$STT_0 = S + SW_1.$$
Now, since X satisfies the property (H), by applying Lemma 3.6, we obtain $T_0 \in \Phi_g(X)$. From the hypothesis $ST \in \Phi_g(X)$, we deduce that $STT_0 \in \Phi_g(X)$. Since SW_1 is a weakly compact operator, it follows that $S \in \Phi_g(X)$. Conversely, suppose that $S \in \Phi_g(X)$ and let $S_0 \in \mathcal{L}(X)$ satisfy $S_0S = I + W$, where $W \in \mathcal{W}(X)$. By the hypothesis $ST \in \Phi_g(X)$ and the fact that $S_0 \in \Phi_g(X)$, we deduce that $S_0ST \in \Phi_g(X)$, and the same holds true for T. □

Remark 3.14. Let X be a non-reflexive Banach space satisfying the property (H). If $F \in \mathcal{F}_g(X)$, then $I + F \in \Phi_g(X)$ and $i_g(I + F) = 0$. Indeed, for all $\mu \in [0, 1]$, $\mu F \in \mathcal{F}_g(X)$, we have $I + \mu F \in \Phi_g(X)$. Now, from the continuity of i_g, we infer that
$$\begin{aligned} \varphi_g : [0, 1] &\to \mathbb{Z}, \\ \mu &\mapsto i_g(I + \mu F), \end{aligned}$$
is continuous on the connected set $[0, 1]$ with discreet values. Then φ_g is constant. Thus, $\varphi_g(0) = \varphi_g(1)$. So, $i_g(I + F) = i_g(I) = 0$.

Our next lemma is an extension of [19, Proposition 3.1].

Lemma 3.15. *Let T and F be two bounded linear operators on a non-reflexive Banach space X satisfying the property* (H)*. If $T \in \Phi_g(X)$, $F \in \mathcal{F}_g(X)$, then*
$$T + F \in \Phi_g(X) \quad \textit{and} \quad i_g(T + F) = i_g(T).$$

Proof. Let $T \in \Phi_g(X)$. Since X satisfies the property (H), then there exist $W_1, W_2 \in \mathcal{W}(X)$ such that $T_0T = I + W_1$ and $TT_0 = I + W_2$ for some $T_0 \in \mathcal{L}(X)$. This leads to
$$T_0(T + F) = I + W_1 + T_0F = I + F_1.$$

Since $\mathcal{F}_g(X)$ is a closed two-sided ideal containing $\mathcal{W}(X)$ (see [4]), we have $F_1 \in \mathcal{F}_g(X)$. It follows from Remark 3.14 that $T_0(T+F) \in \Phi_g(X)$ and $i_g(T_0(T+F)) = 0$. On the other hand, by using Lemma 3.6, we get $T_0 \in \Phi_g(X)$ and $i_g(T_0) = -i_g(T)$. Therefore, $T+F \in \Phi_g(X)$ and $i_g(T+F) = i_g(T)$. □

Definition 3.16. Let X be a non-reflexive Banach space satisfying the property (H) and let T and S be two bounded linear operators on X. We define the generalized Schechter S-essential spectrum of T by

$$\sigma_{Se_5,g}(T) := \mathbb{C} \setminus \{\lambda \in \Phi_{g,T,S} : i_g(\lambda S - T) = 0\}.$$

If $S = I$, then $\sigma_{Se_5,g}(T)$ will simply be denoted by $\sigma_{e_5,g}(T)$.

Theorem 3.17. *Let X be a non-reflexive Banach space having the property* (H). *Let T and S be two bounded linear operators on X. If $\mathbb{C} \setminus \sigma_{Se_4,g}(T)$ is connected and $\rho_S(T)$ is not empty, then*

$$\sigma_{Se_4,g}(T) = \sigma_{Se_5,g}(T).$$

Proof. Since $\sigma_{Se_4,g}(T) \subset \sigma_{Se_5,g}(T)$, we only have to prove the reverse inclusion. Let $\lambda_0 \notin \sigma_{Se_4,g}(T)$. Since $\rho_S(T)$ is not empty, there exists $\lambda_1 \in \mathbb{C}$ such that $\lambda_1 \in \rho_S(T)$. Consequently, $\lambda_1 S - T \in \Phi_g(X)$ and $i_g(\lambda_1 S - T) = 0$. Hence, $\lambda_1 \in \mathbb{C} \setminus \sigma_{Se_5,g}(T)$, which is a subset of $\mathbb{C} \setminus \sigma_{Se_4,g}(T)$. Since $\mathbb{C} \setminus \sigma_{Se_4,g}(T)$ is connected, it follows from Proposition 3.10 that $i_g(\lambda S - T) = 0$ for all $\lambda \in \mathbb{C} \setminus \sigma_{Se_4,g}(T)$. Thus, $\lambda_0 \notin \sigma_{Se_5,g}(T)$. □

Lemma 3.18. *Let A and S be two bounded linear operators on a non-reflexive Banach space X. Suppose that X satisfies the property* (H). *If $B \in \mathcal{F}_g(X)$, then*

$$\sigma_{Se_i,g}(A) = \sigma_{Se_i,g}(A+B), \quad i = 4, 5.$$

Proof. (i) Suppose $B \in \mathcal{F}_g(X)$ and $\lambda \in \mathbb{C}$. Using Lemma 3.15 (i), we obtain $(\lambda S - A) \in \Phi_g(X)$ if and only if $(\lambda S - A - B) \in \Phi_g(X)$. Moreover, $i_g(\lambda S - A) = i_g(\lambda S - A - B)$. Hence, we obtain

$$\sigma_{Se_i,g}(A) = \sigma_{Se_i,g}(A+B), \quad i = 4, 5.$$

This concludes the proof. □

Next, we present a characterization of the generalized S-essential spectrum of the sum of two bounded linear operators acting on a non-reflexive Banach space.

Lemma 3.19 ([12]). *Let X and Y be two Banach spaces and let $Z \subset X$. For an operator $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:*
(i) *T is co-Tauberian.*
(ii) *Every operator $S \in \mathcal{L}(Y, Z)$ is weakly compact whenever ST is weakly compact.*

Theorem 3.20. *Let X be a non-reflexive Banach space having the property H and let S, A and B be three bounded linear operators on X such that $S \neq A$ and $S \neq B$.*
(i) *If $AB \in \mathcal{F}_g(X)$ and $SA = AS$, then*

$$\sigma_{Se_4,g}(A+B) \setminus \{0\} \subset [\sigma_{Se_4,g}(A) \cup \sigma_{Se_4,g}(B)] \setminus \{0\}.$$

Moreover, if $BA \in \mathcal{F}_g(X)$, $SB = BS$ and $S \in \Phi_g(X)$, then we have

$$\sigma_{Se_4,g}(A+B) \setminus \{0\} = [\sigma_{Se_4,g}(A) \cup \sigma_{Se_4,g}(B)] \setminus \{0\}.$$

(ii) *If $AB \in \mathcal{F}_g(X)$, $SA = AS$, $S \in \Phi_g(X)$, and $i_g(S) = 0$, then*

$$\sigma_{Se_5,g}(A+B) \setminus \{0\} \subset [\sigma_{Se_5,g}(A) \cup \sigma_{Se_5,g}(B)] \setminus \{0\}.$$

Moreover, if $BA \in \mathcal{F}_g(X)$, $SB = BS$, $\rho_S(A)$ is not empty, and $\mathbb{C} \setminus \sigma_{Se_4,g}(A)$ is connected, then

$$\sigma_{Se_5,g}(A+B) \setminus \{0\} = [\sigma_{Se_5,g}(A) \cup \sigma_{Se_5,g}(B)] \setminus \{0\}.$$

Proof. For $\lambda \in \mathbb{C}$, we write

$$(\lambda S - A)(\lambda S - B) = AB + \lambda(\lambda S^2 - AS - SB) \tag{3.4}$$

and

$$(\lambda S - B)(\lambda S - A) = BA + \lambda(\lambda S^2 - SA - BS).$$

(i) Let

$$\lambda \notin \sigma_{Se_4,g}(A) \cup \sigma_{Se_4,g}(B) \cup \{0\}.$$

Then $\lambda S - A \in \Phi_g(X)$ and $\lambda S - B \in \Phi_g(X)$. From the fact that X has the property (H), we infer that

$$(\lambda S - A)(\lambda S - B) \in \Phi_g(X).$$

Since $AB \in \mathcal{F}_g(X)$, by using equation (3.4) and Lemma 3.15 (i), we have $(\lambda S^2 - AS - SB) \in \Phi_g(X)$, and hence $S(\lambda S - A - B) \in \Phi_g(X)$. Therefore, $(\lambda S - A - B) \in \Phi_{g+}(X)$. Now by applying Lemma 3.19, it is sufficient to prove that $\lambda S - A - B$ is co-Tauberian. So, let $T \in \mathcal{L}(X)$ such that $T(\lambda S - A - B) \in \mathcal{W}(X)$. On the other hand, we have

$$\overline{\omega}(T(\lambda S - A - B)) > \overline{\omega}(T)\overline{\beta}(\lambda S - A - B).$$

By using Theorem 2.4, we have $\overline{\beta}(\lambda S - A - B) > 0$. Then $\overline{\omega}(T) = 0$. Hence $\lambda S - A - B$ is a co-Tauberian operator and since $\mathcal{R}(\lambda S - A - B)$ is closed, we deduce that $\lambda S - A - B \in \Phi_g(X)$. Thus,

$$\sigma_{Se_4,g}(A + B) \setminus \{0\} \subset [\sigma_{Se_4,g}(A) \cup \sigma_{Se_4,g}(B)] \setminus \{0\}.$$

Conversely, let us suppose that $\lambda \notin \sigma_{Se_4,g}(A + B) \cup \{0\}$. Then $\lambda S - A - B \in \Phi_g(X)$. Since $S \in \Phi_g(X)$, we have $S(\lambda S - A - B) \in \Phi_g(X)$ and $(\lambda S - A - B)S \in \Phi_g(X)$. Since $AB \in \mathcal{F}_g(X)$ and $BA \in \mathcal{F}_g(X)$, we deduce that

$$(\lambda S - A)(\lambda S - B) \in \Phi_g(X) \quad \text{and} \quad (\lambda S - B)(\lambda S - A) \in \Phi_g(X).$$

From Theorem 2.5 (iii), it is clear that $\lambda S - A \in \Phi_g(X)$ and $\lambda S - B \in \Phi_g(X)$. Therefore,

$$\lambda \notin \sigma_{Se_4,g}(A) \cup \sigma_{Se_4,g}(B).$$

(ii) Let

$$\lambda \notin \sigma_{Se_5,g}(A) \cup \sigma_{Se_5,g}(B) \cup \{0\}.$$

Then

$$\lambda S - A \in \Phi_g(X), \quad i_g(\lambda S - A) = 0, \quad \lambda S - B \in \Phi_g(X), \quad i_g(\lambda S - B) = 0.$$

Since X satisfies the property (H), we have $(\lambda S - A)(\lambda S - B) \in \Phi_g(X)$. It is clear that $i_g((\lambda S - A)(\lambda S - B)) = 0$. Moreover, from the hypothesis $AB \in \mathcal{F}_g(X)$, we apply equation (3.4), and Lemma 3.18 (i), which yields that $S(\lambda S - A - B) \in \Phi_g(X)$ and $i_g(S(\lambda S - A - B)) = 0$. Since $S \in \Phi_g(X)$ with $i_g(S) = 0$, by applying Theorem 3.13, we get $\lambda S - A - B \in \Phi_g(X)$ and $i_g(\lambda S - A - B) = 0$. Therefore, $\lambda \notin \sigma_{Se_5,g}(A + B)$, and hence

$$\sigma_{Se_5,g}(A + B) \setminus \{0\} \subset [\sigma_{Se_5,g}(A) \cup \sigma_{Se_5,g}(B)] \setminus \{0\}.$$

Conversely, let us suppose that $\lambda \notin \sigma_{Se_5,g}(A + B) \cup \{0\}$. Then $\lambda S - A - B \in \Phi_g(X)$ and $i_g(\lambda S - A - B) = 0$. Since $AB \in \mathcal{F}_g(X)$, $BA \in \mathcal{F}_g(X)$ and $S \in \Phi_g(X)$, it is easy to prove that $\lambda S - A \in \Phi_g(X)$ and $\lambda S - B \in \Phi_g(X)$. Moreover, by applying equation (4.1) and Lemma 3.18 (i), we get

$$\begin{aligned} i_g((\lambda S - A)(\lambda S - B)) &= i_g(\lambda S - A) + i_g(\lambda S - B) \\ &= i_g(S) + i_g(\lambda S - A - B) \\ &= i_g(\lambda S - A - B) \\ &= 0. \end{aligned}$$

Since $\mathbb{C} \setminus \sigma_{Se_4,g}(T)$ is connected and $\rho_S(A)$ is not empty, Theorem 3.17 yields that $\sigma_{Se_4,g}(A) = \sigma_{Se_5,g}(A)$. Hence, from the fact that $\lambda S - A \in \Phi_g(X)$, we deduce that $i_g(\lambda S - A) = 0$. Therefore, $i_g(\lambda S - B) = 0$. We conclude that $\lambda \notin \sigma_{Se_5,g}(A) \cup \sigma_{Se_5,g}(B)$. □

Now, we establish some results concerning stability in the class of generalized Fredholm operators via the concept of measure of weak noncompactness of De Blasi. First, for $n \in \mathbb{N}^*$, let us consider the following set:

$$\mathcal{W}_n(X) = \{T \in \mathcal{L}(X) : \overline{\omega}((TS)^n) < 1 \text{ for all } S \in \mathcal{L}(X)\}.$$

Remark 3.21. (i) $\mathcal{W}(X) \subset \mathcal{W}_n(X)$ for all $n \in \mathbb{N}^*$.

(ii) If $T \in \mathcal{W}_n(X)$ and $S \in \mathcal{L}(X)$, then $TS \in \mathcal{W}_n(X)$.

(iii) If $T \in \mathcal{W}_n(X)$ and $W \in \mathcal{W}(X)$, then $T + W \in \mathcal{W}_n(X)$. Indeed, for $S \in \mathcal{L}(X)$, we get $((T + W)S)^n = (TS)^n + W'$, where W' is a weakly compact operator. Thus,

$$\overline{\omega}(((T + W)S)^n) = \overline{\omega}((TS)^n) < 1.$$

Remark 3.22. (i) Let X be a non-reflexive Banach space satisfying the property (H) and let $T \in \mathcal{L}(X)$. If $T \in \mathcal{W}_n(X)$, then

$$(I - T) \in \Phi_g(X) \quad \text{and} \quad i_g(I - T) = 0.$$

Indeed, let $T \in \mathcal{W}_n(X)$. Then we obtain that $\overline{\omega}((TS)^n) < 1$ for all $S \in \mathcal{L}(X)$. Thus, we can take $S = I$. So, we get $\overline{\omega}((T)^n) < 1$. Therefore, by using Proposition 2.8, we infer that $(I - T) \in \Phi_g(X)$. Moreover, for $\lambda \in [0, 1]$, we get $\overline{\omega}((\lambda T)^n) < 1$. Then as above, $(I - \lambda T) \in \Phi_g(X)$. Now, by the continuity of the generalized index on $\Phi_g(X)$, we deduce that $i_g(I - T) = i_g(I) = 0$.

(ii) We have $\mathcal{W}_n(X) \subset \mathcal{WDC}(X)$ for all $n \in \mathbb{N}^*$, where $\mathcal{WDC}(X)$ is the set of weakly demicompact operators. Indeed, let $T \in \mathcal{W}_n(X)$. Then we get $\overline{\omega}(T^n) < 1$. By using [18, Theorem 2.1], we infer that $\lambda T \in \mathcal{WDC}(X)$ for all $\lambda \in [0, 1]$.

Proposition 3.23. *Let X be a non-reflexive Banach space satisfying the property* (H) *and let $T \in \mathcal{L}(X)$. Then the following statements hold:*

(i) $\bigcup_{n\in\mathbb{N}^*} \mathcal{W}_n(X) \subset \mathcal{F}_g(X)$.

(ii) *If $J \in \bigcup_{n\in\mathbb{N}^*} \mathcal{W}_n(X)$, then*

$$\sigma_{Se_i,g}(T) = \sigma_{Se_i,g}(T + J), \quad i = 4, 5.$$

Proof. (i) For $n \in \mathbb{N}^*$, let $J \in \mathcal{W}_n(X)$ and $T \in \Phi_g(X)$. Then, from [24, Theorem 5.10], there exist $T_0 \in \mathcal{L}(X)$ and $W \in \mathcal{W}(X)$ such that

$$TT_0 = I + W.$$

This leads to

$$(T + J)T_0 = I + W + JT_0 = I + F,$$

where $F = W + JT_0$. Since $J \in \mathcal{W}_n(X)$ and $T_0 \in \mathcal{L}(X)$, by applying Remark 3.21 (ii), we infer that $JT_0 \in \mathcal{W}_n(X)$. This implies that $F \in \mathcal{W}_n(X)$ (see Remark 3.21 (iii)). Now, it follows from Remark 3.22 (i) that $(T + J)T_0 \in \Phi_g(X)$ and $i_g((T + J)T_0) = 0$. From the fact that $T_0 \in \Phi_g(X)$ and $i_g(T_0) = -i_g(T)$, we deduce that $T + J \in \Phi_g(X)$ and $i_g(T + J) = i_g(T)$.

(ii) This assumption is a consequence of (i) and Lemma 3.18. □

In the next result, a refinement of Proposition 2.7 is presented.

Theorem 3.24. *Let X be a non-reflexive Banach space having the property* (H) *and let T and S be two bounded linear operators on X. Then*

$$\sigma_{Se_1,g}(T) = \bigcap_{J\in\mathcal{W}_n(X)} \sigma_{Se_4,g}(T + J).$$

Proof. Taking into account that $\mathcal{W}(X) \subset \mathcal{W}_n(X)$, we deduce by Proposition 2.7 that

$$\bigcap_{J\in\mathcal{W}_n(X)} \sigma_{Se_4,g}(T + J) \subset \sigma_{Se_1,g}(T).$$

Conversely, let

$$\lambda \notin \bigcap_{J\in\mathcal{W}_n(X)} \sigma_{Se_4,g}(T + J).$$

Then there exists $J \in \mathcal{W}_n(X)$ such that $\lambda \notin \sigma_{Se_4,g}(T+J)$. Hence, $(\lambda S - T - J) \in \Phi_g(X)$. Since X satisfies property (H) and $J \in \mathcal{W}_n(X)$, by using Proposition 3.23, we deduce that $(\lambda S - T) \in \Phi_g(X)$. Consequently, $\lambda \notin \sigma_{Se_1,g}(T)$, which completes the proof. □

We now prove that the generalized Wolf essential spectrum satisfies the spectral mapping theorem. However, in general it is not true for the generalized Schechter essential spectrum.

Theorem 3.25. *Let X be a non-reflexive Banach space having the property* (H) *and let f be a complex-valued function which is locally holomorphic on an open set containing the spectrum of a linear operator $T \in \mathcal{L}(X)$. Then the following assertions hold:*
(i) $\sigma_{e_4,g}(f(T)) = f(\sigma_{e_4,g}(T))$.
(ii) $\sigma_{e_5,g}(f(T)) \subset f(\sigma_{e_5,g}(T))$.

Proof. (i) Since X satisfies the property (H), we have that $\Phi_g(X)$ is an F-semigroup in $\mathcal{L}(X)$ (see Example 3.4). Hence, by applying [13, Theorem 1] in the case of the F-semigroup $\Phi_g(X)$, we deduce that $\sigma_{e_4,g}(f(T)) = f(\sigma_{e_4,g}(T))$.

(ii) For $n \in \mathbb{N}$, we define

$$\Phi_{g,n}(X) := \{T \in \Phi_g(X) : i_g(T) = n\}$$

and

$$F_n := \{\lambda \in \mathbb{C} : \lambda - T \in \Phi_{g,n}(X)\}.$$

Observe that

$$\sigma_{e_5,g}(T) = \sigma_{e_4,g}(T) \cup \Big(\bigcup_{n \neq 0} F_n\Big). \tag{3.5}$$

Now, let $\mu \notin f(\sigma_{e_5,g}(T))$. Then $\mu - f(\lambda)$ has no zeros on $\sigma_{e_5,g}(T)$, and in particular has no zeros on $\sigma_{e4,g}(T)$. From [13, Theorem 1], applied to the F-semigroup $\Phi_g(X)$ with generalized index i_g, we conclude that $\mu - f(T) \in \Phi_g(X)$ and

$$i_g(\mu - f(T)) = \sum_{n \neq 0} n\alpha_n,$$

where α_n is the number of isolated zeros of $\mu - f(\lambda)$ on F_n, counted according to their multiplicities. From equation (3.5), we deduce that $\alpha_n = 0$ for every $n \neq 0$. Therefore, $i_g(\mu - f(T)) = 0$, and hence $\mu \notin \sigma_{e_5,g}(f(T))$. □

Following [15], we show now that the generalized Schechter essential spectrum satisfies the spectral mapping theorem in a special case.

Theorem 3.26. *Let X be a non-reflexive Banach space satisfying the property* (H) *and let T_1 and T_2 be two bounded linear operators on X such that $T_1 - T_2 \in \mathcal{W}_n(X)$ for some $n \in \mathbb{N}^*$. If $\sigma_{e_5,g}(T_1) \subset \sigma_{e_4,g}(T_1)$ and f is a complex-valued function locally holomorphic on $\sigma(T_1) \cup \sigma(T_2) \cup \{\infty\}$, then*

$$f(\sigma_{e_5,g}(T_k)) = \sigma_{e_5,g}(f(T_k)), \quad k = 1, 2.$$

Proof. For $k = 1$, the result follows from the hypothesis $\sigma_{e_4,g}(T_1) = \sigma_{e_5,g}(T_1)$ and Theorem 3.25 (i). Now, for $k = 2$, Theorem 3.25 (ii) shows that $\sigma_{e_5,g}(f(T_2)) \subset f(\sigma_{e_5,g}(T_2))$. So, we only have to prove that

$$f(\sigma_{e_5,g}(T_2)) \subset \sigma_{e_5,g}(f(T_2)).$$

Let $\lambda \in f(\sigma_{e_5,g}(T_2))$. Then there exists $\mu \in \sigma_{e_5,g}(T_2)$ such that $\lambda = f(\mu)$. Using the hypothesis $T_1 - T_2 \in \mathcal{W}_n(X)$ and Proposition 3.23, we infer that $\sigma_{e_5,g}(T_2) = \sigma_{e_5,g}(T_1)$. From the hypothesis $\sigma_{e_5,g}(T_1) \subset \sigma_{e_4,g}(T_1)$, we have $\sigma_{e_4,g}(T_1) = \sigma_{e_5,g}(T_1)$, which implies that

$$\mu \in \sigma_{e_4,g}(T_1) = \sigma_{e_4,g}(T_2).$$

Next, making use of the spectral mapping theorem of the generalized Wolf essential spectrum, we conclude that $f(\mu) \in \sigma_{e_4,g}(f(T_2))$. Since $\sigma_{e_4,g}(f(T_2)) \subset \sigma_{e_5,g}(f(T_2))$, we have $f(\mu) \in \sigma_{e_5,g}(f(T_2))$. This ends the proof. □

Now, let $\mathcal{P}(X)$ denote the set

$$\mathcal{P}(X) = \{F \in \mathcal{L}(X) : \text{there exists a nonzero complex polynomial } p(\cdot) \text{ satisfying } p(F) \in \mathcal{W}(X)\}.$$

If $F \in \mathcal{P}(X)$, then the nonzero polynomial $p(\cdot)$ of least degree and leading coefficient 1 such that $p(F) \in \mathcal{W}(X)$ will be called the minimal polynomial of F. We denote by $\mathcal{PW}(X)$ the subset of $\mathcal{P}(X)$ defined by

$$\mathcal{PW}(X) = \{F \in \mathcal{P}(X) \text{ such that the minimal polynomial } p(\cdot) \text{ of } F \text{ satisfies } p(-1) \neq 0\}.$$

As an immediate consequence of the spectral mapping theorem of the generalized Wolf essential spectrum, we have the following lemma.

Lemma 3.27. *Let X be a non-reflexive Banach space having the property* (H). *If $F \in \mathcal{PW}(X)$, then $I + F \in \Phi_g(X)$.*

Proof. Let $p(\cdot)$ be the minimal polynomial of F. Since $p(F) \in \mathcal{W}(X)$, we have $\sigma_{e_4,g}(p(F)) = \{0\}$. So, by the hypothesis $p(-1) \neq 0$, we infer that $p(-1) \notin \sigma_{e_4,g}(p(F))$. Hence, by applying Theorem 3.25 (i), we deduce that $p(-1) \notin p(\sigma_{e_4,g}(F))$. Therefore, $-1 \notin \sigma_{e_4,g}(F)$. □

Theorem 3.28. *Let X be a non-reflexive Banach space having the property* (H) *and let T and S be two bounded linear operators on X. Suppose that there exist $T_0, S_0 \in \mathcal{L}(X)$ and $J_1, J_2 \in \mathcal{PW}(X)$ satisfying*

$$\begin{aligned} TT_0 &= I + J_1, \\ SS_0 &= I + J_2. \end{aligned} \tag{3.6}$$

If $0 \in \Phi_{g,T} \cap \Phi_{g,S}$, $T_0 - S_0 \in \mathcal{W}_n(X)$ and $J_1 - J_2 \in \mathcal{W}(X)$, then

$$\sigma_{e_4,g}(T) = \sigma_{e_4,g}(S).$$

Moreover, if $i_g(T) = i_g(S) = 0$ and $\sigma_{e_5,g}(J_1) \subset \sigma_{e_4,g}(J_1)$, then we have

$$\sigma_{e_5,g}(T) = \sigma_{e_5,g}(S). \tag{3.7}$$

Proof. (i) Let λ be a complex number. It follows from equations (3.6) and (3.7) that

$$(\lambda - T)T_0 - (\lambda - S)S_0 = \lambda(T_0 - S_0) - (J_1 - J_2). \tag{3.8}$$

If $\lambda \notin \sigma_{e_4,g}(S)$, then $\lambda - S \in \Phi_g(X)$. Since $J_2 \in \mathcal{PW}(X)$, applying Lemma 3.27, we get $I + J_2 \in \Phi_g(X)$. So, it follows from equation (3.7) that $SS_0 \in \Phi_g(X)$. As $S \in \Phi_g(X)$, it follows from Theorem 3.13 that $S_0 \in \Phi_g(X)$. Therefore, by using [5, Theorem 3.11 (i)], we infer that $(\lambda - S)S_0 \in \Phi_g(X)$. If $T_0 - S_0 \in \mathcal{W}_n(X)$, then equation (3.8) and Remark 3.21 together with Proposition 3.23 imply that $(\lambda - T)T_0 \in \Phi_g(X)$. Thus, $\lambda - T \in \Phi_g(X)$, which implies that $\sigma_{e_4,g}(T) \subset \sigma_{e_4,g}(S)$. The opposite inclusion follows by symmetry.

To prove equation (3.7), let $\lambda \notin \sigma_{e_5,g}(S)$. Then $\lambda - S \in \Phi_g(X)$ and $i_g(\lambda - S) = 0$. Since $J_2 \in \mathcal{PW}(X)$, there exists a nonzero complex polynomial $p(\cdot)$ such that $p(J_2) \in \mathcal{W}(X)$. Then $\sigma_{e_5,g}(p(J_2)) = \{0\}$. From the fact that $p(-1) \neq 0$, we deduce that $p(-1) \notin \sigma_{e_5,g}(p(J_2))$. Now, since

$$J_1 - J_2 \in \mathcal{W}_n(X) \quad \text{and} \quad \sigma_{e_5,g}(J_1) \subset \sigma_{e_4,g}(J_1),$$

by applying Theorem 3.26, we have

$$p(\sigma_{e_5,g}(J_2)) = \sigma_{e_5,g}(p(J_2)),$$

and hence $p(-1) \notin p(\sigma_{e_5,g}(J_2))$. Therefore, $I + J_2 \in \Phi_g(X)$ and $i_g(I - J_2) = 0$. It follows from equation (3.7) that $(\lambda - S)S_0 \in \Phi_g(X)$ and $i_g(\lambda - S)S_0) = 0$. Since $T_0 - S_0 \in \mathcal{W}_n(X)$, by using equation (3.8), Remark 3.21 and Proposition 3.23, we infer that

$$(\lambda - T)T_0 \in \Phi_g(X) \quad \text{and} \quad i_g((\lambda - T)T_0) = 0.$$

Hence, $\lambda - T \in \Phi_g(X)$ and $i_g(\lambda - T) = 0$, and therefore $\sigma_{e_5,g}(T) \subset \sigma_{e_5,g}(S)$. The opposite inclusion follows by symmetry. □

4 Generalized Fredholm theory for operator matrices

We begin this section by obtaining some results which extend certain theorems in [14] to generalized Fredholm operators.

Lemma 4.1. *Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X)$. Consider the 2×2 operator matrix*

$$M_A = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

Then $A \in \Phi_g(X)$ if and only if $M_A \in \Phi_g(X \times Y)$.

Proof. Suppose that $A \in \Phi_g(X)$. First, recall that a finite product of reflexive spaces is also reflexive. Since $\mathcal{N}(M_A) = \mathcal{N}(A) \times \{0\}$, we conclude that $\mathcal{N}(M_A)$ is reflexive. On the other hand, obviously, $\mathcal{R}(M_A) = \mathcal{R}(A) \times Y$. Since $X \times Y/\mathcal{R}(A) \times Y$ is isomorphic to $X/\mathcal{R}(A) \times \{\overline{0}\}$, it follows that $X \times Y/\mathcal{R}(M_A)$ is reflexive. Moreover, $\mathcal{R}(M_A)$ is closed, and hence $M_A \in \Phi_g(X \times Y)$. □

Remark 4.2. By using the same reasoning as in the proof of Lemma 4.1, we prove that $B \in \Phi_g(Y)$ if and only if

$$M_B = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \in \Phi_g(X \times Y).$$

Theorem 4.3. *Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. Consider the 2×2 operator matrix*

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad \textit{where } C \in \mathcal{L}(Y \times X).$$

Suppose that $X \times Y$ is not reflexive.
(i) *If $X \times Y$ satisfies the property* (H_1), *$A \in \Phi_{g+}(X)$ and $B \in \Phi_{g+}(Y)$, then*

$$M_C \in \Phi_{g+}(X \times Y)$$

for every $C \in \mathcal{L}(Y \times X)$.
(ii) *If $(X \times Y)^*$ satisfies the property* (H_1), *$A \in \Phi_{g-}(X)$ and $B \in \Phi_{g-}(Y)$, then*

$$M_C \in \Phi_{g-}(X \times Y)$$

for every $C \in \mathcal{L}(Y \times X)$.
(iii) *If $X \times Y$ satisfies the property* (H), *$A \in \Phi_g(X)$ and $B \in \Phi_g(Y)$, then*

$$M_C \in \Phi_g(X \times Y)$$

for every $C \in \mathcal{L}(Y \times X)$.

Proof. (i) Suppose that $A \in \Phi_{g+}(X)$ and $B \in \Phi_{g+}(X)$. Then $\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ are both in $\Phi_{g+}(X \times Y)$. Now we can write

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}. \tag{4.1}$$

Since $X \times Y$ satisfies the property (H_1) and $\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ is invertible for every $C \in \mathcal{L}(Y, X)$, it follows from Theorem 2.6 (iii) that $M_C \in \Phi_{g+}(X \times Y)$.

(ii) The proof of assumption (ii) is similar to assumption (i), it suffices to replace Theorem 2.6 (iii) by Theorem 2.6 (iv).

(iii) It follows from assertions (i) and (ii). □

Remark 4.4. Let us consider the 2×2 operator matrix

$$M_D = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}, \quad \text{where } D \in \mathcal{L}(X \times Y).$$

In the same way as in the proof of Theorem 4.3, we prove that the following assertions hold:
(i) If $X \times Y$ satisfies the property (H_1) and if $A \in \Phi_{g+}(X)$ and $B \in \Phi_{g+}(Y)$, then

$$M_D \in \Phi_{g+}(X \times Y)$$

for every $D \in \mathcal{L}(X \times Y)$.

(ii) If $(X \times Y)^*$ satisfies the property (H_1) and if $A \in \Phi_{g-}(X)$ and $B \in \Phi_{g-}(Y)$, then

$$M_D \in \Phi_{g-}(X \times Y)$$

for every $D \in \mathcal{L}(X \times Y)$.

(iii) If $(X \times Y)$ satisfies the property (H) and if $A \in \Phi_g(X)$ and $B \in \Phi_g(Y)$, then

$$M_D \in \Phi_g(X \times Y)$$

for every $D \in \mathcal{L}(X \times Y)$.

Proposition 4.5. *Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. Consider the 2×2 operator matrix*

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad \text{where } C \in \mathcal{L}(Y \times X).$$

Suppose that $X \times Y$ is not reflexive.

(i) *If $X \times Y$ satisfies the property* (H_1) *and $M_C \in \Phi_{g+}(X \times Y)$, then $A \in \Phi_{g+}(X)$.*

(ii) *If $X \times Y$ and $(X \times Y)^*$ satisfy the property* (H_1) *and $M_C \in \Phi_{g-}(X \times Y)$, then $B \in \Phi_{g-}(Y)$.*

Proof. It follows immediately from equation (4.1) and Theorem 2.5. □

Remark 4.6. (i) Suppose that $X \times Y$ satisfies the property (H_1). If $M_C \in \Phi_g(X \times Y)$, then $A \in \Phi_{g+}(X)$ and $B \in \Phi_{g-}(Y)$.

(ii) Similarly, we show that if $X \times Y$ satisfies the property (H_1) and the operator

$$\begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \in \Phi_g(X \times Y)$$

for some $D \in \mathcal{L}(X \times Y)$, then $A \in \Phi_{g-}(X)$ and $B \in \Phi_{g+}(Y)$.

Theorem 4.7. *Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. Suppose that $X \times Y$ is not reflexive. Then for every $C \in \mathcal{L}(Y, X)$, the following assertions hold:*

(i) *If $X \times Y$ satisfies the property* (H), *then*

$$\sigma_{e_4,g}(M_C) \subset \sigma_{e_4,g}(A) \cup \sigma_{e_4,g}(B).$$

Moreover, if $\sigma_{e_4,g}(A) \subset \sigma_{e_4,g}(M_C)$ or $\sigma_{e_4,g}(B) \subset \sigma_{e_4,g}(M_C)$, then

$$\sigma_{e_4,g}(M_C) = \sigma_{e_4,g}(A) \cup \sigma_{e_4,g}(B).$$

(ii) *If $X \times Y$ satisfies the property* (H_1), *then*

$$\sigma_{e_1,g}(M_C) \subset \sigma_{e_1,g}(A) \cup \sigma_{e_1,g}(B).$$

Proof. (i) Let $\lambda \notin [\sigma_{e_4,g}(A) \cup \sigma_{e_4,g}(B)]$. Then $\lambda - A$ and $\lambda - B$ are both generalized Fredholm operators. Since $X \times Y$ satisfies the property (H), by using Theorem 4.3 (iii), we infer that $\lambda - M_C \in \Phi_g(X \times Y)$, which implies that $\lambda \notin \sigma_{e_4,g}(M_C)$.

To prove the opposite inclusion, assume that $\sigma_{e_4,g}(A) \subset \sigma_{e_4,g}(M_C)$ or $\sigma_{e_4,g}(B) \subset \sigma_{e_4,g}(M_C)$, and suppose that $\sigma_{e_4,g}(M_C) \neq \sigma_{e_4,g}(A) \cup \sigma_{e_4,g}(B)$. Then there exists λ such that

$$\lambda \in [\sigma_{e_4,g}(A) \cup \sigma_{e_i,g}(B)] \setminus \sigma_{e_4,g}(M_C).$$

Observe that if $X \times Y$ satisfies the property (H) and $M_C \in \Phi_g(X \times Y)$, then $A \in \Phi_{g+}(X)$ and $B \in \Phi_{g-}(X)$. From these observations, we notice that if M_C is generalized Fredholm, then A is generalized Fredholm if and only if B is generalized Fredholm. Hence $\lambda \in \sigma_{e_4,g}(A) \cap \sigma_{e_i,g}(B)$, a contradiction. Therefore,

$$\sigma_{e_4,g}(M_C) = \sigma_{e_4,g}(A) \cup \sigma_{e_4,g}(B).$$

(ii) Let $\lambda \notin [\sigma_{e_1,g}(A) \cup \sigma_{e_1,g}(B)]$. Then $\lambda - A \in \Phi_{g+}(X)$ and $\lambda - B \in \Phi_{g+}(Y)$. Since $X \times Y$ satisfies the property (H_1), it follows from Theorem 4.3 (i) that $\lambda - M_C \in \Phi_{g+}(X \times Y)$, and therefore $\lambda \notin \sigma_{e_1,g}(M_C)$. □

5 Conclusion

Our work is motivated by a larger class than that of Fredholm operators, the class of so-called generalized Fredholm operators, which was introduced by Yang in [24] in order to extend some perturbation results. The generalized Fredholm properties may not be stable under perturbation, and hence some conditions on the space on which the operators are defined have to be considered. In fact, we are interested in the study of the stability of some generalized S-essential spectra of bounded linear operators acting on non-reflexive Banach spaces having the property (H) under the class of generalized Fredholm perturbations, for example, the generalized Schechter S-essential spectrum

$$\sigma_{Se_5,g}(T) = \mathbb{C} \setminus \{\lambda \in \Phi_{g,T,S} : i_g(\lambda S - T) = 0\},$$

where $i_g(\cdot)$ is an extended index defined for generalized Fredholm operators. The results investigated in the present paper extend those given in [14] and generalize some of the results reviewed in [22]. Furthermore, a generalized Fredholm theory for 2×2 block operator matrices acting on a non-reflexive product of two Banach spaces is developed. It is possible that such investigations may be extended to closed linear operators.

References

[1] P. Aiena, *Fredholm and Local Spectral Theory, with Applications to Multipliers*, Kluwer Academic, Dordrecht, 2004.
[2] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Grad. Texts in Math. 233, Springer, New York, 2006.
[3] K. Astala, On measures of noncompactness and ideal variations in Banach spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math. Diss.* **29** (1980), 1–42.
[4] A. Azzouz, M. Beghdadi and B. Krichen, g-Riesz operators and their spectral properties, *Adv. Oper. Theory* **7** (2022), no. 2, Paper No. 18.
[5] A. Azzouz, M. Beghdadi and B. Krichen, Generalized relative essential spectra, *Filomat* **36** (2022), no. 8, 2657–2673.
[6] J. Banaś and J. Rivero, On measures of weak noncompactness, *Ann. Mat. Pura Appl. (4)* **151** (1988), 213–224.
[7] F. S. De Blasi, On a property of the unit sphere in a Banach space, *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.)* **21(69)** (1977), no. 3–4, 259–262.
[8] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Pure Appl. Math. 7, Interscience, New York, 1958.
[9] G. Emmanuele, Measure of weak noncompactness and fixed point theorems, *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.)* **25(73)** (1981), no. 4, 353–358.
[10] I. C. Gohberg, A. S. Markus and I. A. Feldman, Normally solvable operators and ideals associated with them, *Amer. Math. Soc. Transl. Ser.* **2** (1967), no. 61, 63–84.
[11] S. Goldberg, *Unbounded Linear Operators: Theory and Applications*, McGraw-Hill, New York, 1966.
[12] M. González and A. Martínez-Abejón, *Tauberian Operators*, OOper. Theory Adv. Appl. 194, Birkhäuser, Basel, 2010.
[13] B. Gramsch and D. Lay, Spectral mapping theorems for essential spectra, *Math. Ann.* **192** (1971), 17–32.
[14] A. Jeribi, *Spectral Theory and Applications of Linear Operators and Block Operator Matrices*, Springer, Cham, 2015.
[15] A. Jeribi and B. Krichen, *Nonlinear Functional Analysis in Banach Spaces and Banach Algebras. Fixed Point Theory under Weak Topology for Nonlinear Operators and Block Operator Matrices with Applications*, Monogr. Res. Notes Math., CRC Press, Boca Raton, 2016,
[16] A. Jeribi and M. Mnif, Fredholm operators, essential spectra and application to transport equations, *Acta Appl. Math.* **89** (2005), no. 1–3, 155–176.
[17] T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren Math. Wiss. 132, Springer, New York, 1966.
[18] B. Krichen and D. O'Regan, Weakly demicompact linear operators and axiomatic measures of weak noncompactness, *Math. Slovaca* **69** (2019), no. 6, 1403–1412.
[19] K. Latrach and A. Dehici, Fredholm, semi-Fredholm perturbations, and essential spectra, *J. Math. Anal. Appl.* **259** (2001), no. 1, 277–301.
[20] K. Latrach and A. Jeribi, Some results on Fredholm operators, essential spectra, and application, *J. Math. Anal. Appl.* **225** (1998), no. 2, 461–485.
[21] A. Peł czyński, Projections in certain Banach spaces, *Studia Math.* **19** (1960), 209–228.
[22] M. Schechter, *Principles of Functional Analysis*, Academic Press, New York, 1971.
[23] C. Tretter, *Spectral Theory of Block Operator Matrices and Applications*, Imperial College, London, 2008.
[24] K. W. Yang, The generalized Fredholm operators, *Trans. Amer. Math. Soc.* **216** (1976), 313–326.