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Invariance of essential spectrum by means of generalized weak demicompactness

Abstract: In this paper, we determine the left (resp. right) spectrum of the sum of two bounded linear operators via the class of generalized weakly demicompact operator. Moreover, an investigation of the left (resp. right) essential spectrum of the closure of matrix operator is given.

Keywords: Generalized weakly demicompact operators, matrix operators, essential spectra

MSC 2010: 47A53, 47A10

1 Introduction

During the past years, the invariance of the essential spectra under perturbation was an interesting problem in the study of essential spectra of linear operators. Motivated by the notion of generalized weakly demicompact operator (in short GWDC) introduced by I. Ferjani et al. in [5], we gather a characterization of the left and right spectrum of the sum of two operators. This notion generalized the known class of demicompact operators which was introduced by W. V. Petryshyn in 1966. Later W. V. Petryshyn [17] and W. Y. Akashi [1] used the demicompactness to provide some results on Fredholm operators. Later W. Chaker et al. [4] investigated the essential spectra of densely defined operators involving this class. In [11], B. Krichen generalized this concept by introducing the relative demicompactness. In 2018, B. Krichen et al. [12] extended the above results to the weak topology. Moreover, in [13], the same authors used a measure of weak noncompactness to characterize the weakly demicompact operators. Recently, in 2020, I. Ferjani et al. investigated the so-called relative generalized weak demicompactness [6] and in [7] the same authors determined the relative essential spectra of 3×3 block matrix operator. In the theory of unbounded matrix operators, we mention the work [19, 2]. In 2018, F. Ben Brahim et al. ensured the demicompactness of the closure of matrix operator with domain consisting of vectors satisfying some conditions. In 2019, A. Jeribi et al. [9] determined the essential spectra of 2×2 block matrix operator involving demicompactness. For their study they used the tool of the Frobenius–Schur factorization which was recognized by R. Nagel in [15, 16]. Furthermore, the same authors in [10] studied the γ-diagonally dominant matrix operators.

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2 Preliminary

In this section we recall some definitions and notations that we will need in the sequel. Let X and Y be two Banach spaces. We designate by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$), the set of all closed densely defined linear operators (resp. the set of all bounded linear operators) acting from X into Y and by $\mathcal{K}(X,Y)$ the subset of compact operators of $\mathcal{L}(X,Y)$. For $T \in \mathcal{C}(X,Y)$, we denote by $\mathcal{N}(T)$ (resp. $\mathcal{R}(T)$), the kernel (resp. the range) of T. The dimension of the kernel is denoted by $\alpha(T)$ and the codimension of the range is denoted by $\beta(T)$.

The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\};$$
$$\Phi(X,Y) = \Phi_-(X,Y) \cap \Phi_+(X,Y)$$

designates the set of Fredholm operators.

The set of left Fredholm operators is defined as

$$\Phi_l(X,Y) = \{T \in \mathcal{L}(X,Y) : \mathcal{R}(T) \text{ is a closed and complemented subspace of } Y \text{ and } \alpha(T) < \infty\}.$$

The set of right Fredholm operators is defined as

$$\Phi_r(X,Y) = \{T \in \mathcal{L}(X,Y) : \mathcal{N}(T) \text{ is a closed and complemented subspace of } X \text{ and } \beta(T) < \infty\}.$$

We have the following inclusions:

$$\Phi(X,Y) \subset \Phi_l(X,Y) \quad \text{and} \quad \Phi(X,Y) \subset \Phi_r(X,Y).$$

For an operator $T \in \Phi(X,Y)$, the number $i(T) = \alpha(T) - \beta(T)$ is called the index of T. A complex number λ is in Φ_T, Φ_l, or Φ_r if $\lambda - T$ is in $\Phi(X,Y)$, $\Phi_l(X,Y)$, or $\Phi_r(X,Y)$, respectively. If $X = Y$, the above sets will be denoted by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_l(X)$, and $\Phi_r(X)$. If $T \in \mathcal{C}(X)$, the resolvent set (resp. the spectrum) of T is denoted by $\rho(T)$ (resp. $\sigma(T)$).

Now, a bounded operator T is called a Fredholm perturbation (in short FP) if $S+T \in \Phi(X,Y)$ whenever $S \in \Phi(X,Y)$.

Theorem 2.1 ([18, 14]). *Let X, Y, and Z be Banach spaces. Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$.*

(i) *If* $T \in \Phi_+(X,Y)$ *and* $S \in \Phi_+(Y,Z)$*, then* $ST \in \Phi_+(X,Z)$.
(ii) *If* $T \in \Phi(X,Y)$ *and* $S \in \Phi(Y,Z)$*, then* $ST \in \Phi(X,Z)$*,* $i(TS) = i(T) + i(S)$.
(iii) *If* $X = Y = Z$*,* $TS \in \Phi(X)$ *and* $ST \in \Phi(X)$*, then* $T \in \Phi(X)$ *and* $S \in \Phi(X)$.

Definition 2.2. Let X and Y be two Banach spaces.
(i) An operator $T \in \mathcal{C}(X,Y)$ is said to have a left Fredholm inverse if there exists $T_l \in \mathcal{L}(Y,X_T)$ such that $I_{X_T} - T_l\widehat{T} \in \mathcal{K}(X_T)$.
(ii) An operator $T \in \mathcal{C}(X,Y)$ is said to have a right Fredholm inverse if there exists $T_r \in \mathcal{L}(Y,X_T)$ such that $I_Y - \widehat{T}T_r \in \mathcal{K}(Y)$.

We define the sets $\mathbb{F}^l_T(X)$ and $\mathbb{F}^r_T(X)$ as follows:

$$\mathbb{F}^l_T(X) = \{T_l \in \mathcal{L}(X,X_T) \text{ such that } T_l \text{ is a left Fredholm inverse of } T\},$$

$$\mathbb{F}^r_T(X) = \{T_r \in \mathcal{L}(X,X_T) \text{ such that } T_r \text{ is a right Fredholm inverse of } T\}.$$

Theorem 2.3 ([2]). *Let the matrix operator* L_0 *satisfy the conditions* $(H1)$–$(H3)$ *and the lineal* $\mathcal{D}(B) \cap \mathcal{D}(D)$ *be dense in* X*. Then* L_0 *is closable and its closure* L *is given by*

$$L = \mu - \begin{bmatrix} I & 0 \\ F(\mu) & I \end{bmatrix} \begin{bmatrix} \mu - A & 0 \\ 0 & \mu - S(\mu) \end{bmatrix} \begin{bmatrix} I & G(\mu) \\ 0 & I \end{bmatrix}. \tag{2.1}$$

Remark 2.4. Let $\beta \in \mathbb{C}$. It follows from Eq. (2.1) that

$$\begin{aligned}\beta - L &= \begin{bmatrix} I & 0 \\ F(\mu) & I \end{bmatrix} \begin{bmatrix} \beta - A & 0 \\ 0 & \beta - S(\mu) \end{bmatrix} \begin{bmatrix} I & G(\mu) \\ 0 & I \end{bmatrix} - (\beta - \mu)\mathbb{T}(\mu) \\ &:= \mathbb{U}\mathbb{V}(\beta)\mathbb{W} - (\beta - \mu)\mathbb{T}(\mu),\end{aligned}$$

where $\mathbb{T}(\mu) = \left[\begin{smallmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{smallmatrix}\right]$.

We deal in this paper with the following essential spectra:

$$\sigma_l(T) = \{\lambda \in \mathbb{C} \text{ such that} \lambda - T \notin \Phi_l(X)\},$$

$$\sigma_r(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_r(X)\},$$

$$\sigma_{e_{wl}}(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_l(X) \text{ and } i(\lambda - T) \leq 0\},$$

$$\sigma_{e_{wr}}(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_l(X) \text{ and } i(\lambda - T) \geq 0\}.$$

3 Main results

Theorem 3.1. *Let* X *be a Banach space,* $F \in \mathcal{L}(X)$*, and* $G \in \mathcal{L}(X)$*. Suppose that there is a subset* Ω *of the complex field such that* Ω *is finite and* $0 \in \Omega$*. If for every* $\beta \in \Phi_{l(F+G)} \backslash \Omega$, *there is $R_{\beta l}, S_{\beta l} \in \mathbb{F}^{l}_{\beta I-F-G}$ such that $-\mu FGR_{\beta l}$ and $-\mu GFS_{\beta l}$ are GWDC with generalized set Ω, then*

$$[\sigma_l(F) \cup \sigma_l(G)]\backslash\Omega \subset [\sigma_l(F+G)]\backslash\Omega.$$

Proof. Let us consider $\beta \in \mathbb{C}\backslash\Omega$. Assume that there are $R_{\beta l}, S_{\beta l} \in \mathbb{F}^{l}_{\beta I-F-G}$. Thus, we infer that there exists $K \in \mathcal{K}(X)$ such that $R_{\beta l}(\beta I - F - G) = I - K$. Further, we obtain that

$$\begin{aligned}(\beta I - F)(\beta I - G) &= \beta(\beta I - F - G) + FG \\ &= \beta(\beta I - F - G) + FGR_{\beta l}(\beta I - F - G) + FGK.\end{aligned}$$

So, we get the following equality:

$$(\beta I - F)(\beta I - G) = (\beta I + FGR_{\beta l})(\beta I - F - G) + FGK. \tag{3.1}$$

Also, we can write

$$(\beta I - G)(\beta I - F) = (\beta I + GFS_{\beta l})(\beta I - F - G) + GFK. \tag{3.2}$$

Let $\beta \notin \sigma_l(F+G) \cup \Omega$, which implies that $\beta I - F - G \in \Phi_l(X)$.

Now, let us consider $R_{\beta l}, S_{\beta l} \in \mathbb{F}^{l}_{\beta I-F-G}$ such that $-\mu FGR_{\beta l}$ and $-\mu GFS_{\beta l}$ are GWDC with generalized set Ω. Consequently, in view of Theorem 3.2 in [5], we infer that $(\beta I + FGR_{\beta l})$ and $(\beta I + GFS_{\beta l})$ are Fredholm operators on X, for all $\beta \in \mathbb{C}\backslash\Omega$. Now, the use of Eqs. (3.1) and (3.2) and Theorem 2.1 allows us to conclude that $(\beta I - F)(\beta I - G) \in \Phi_l(X)$ and $(\beta I - G)(\beta I - F) \in \Phi_l(X)$. Thus, we deduce that $\beta I - F \in \Phi_l(X)$ and $\beta I - G \in \Phi_l(X)$, for all $\beta \in \mathbb{C}\backslash\Omega$. □

Theorem 3.2. *Let X be a Banach space, $F \in \mathcal{L}(X)$, and $G \in \mathcal{L}(X)$. Suppose that there is a subset Ω of the complex field such that Ω is finite and $0 \in \Omega$. If for every $\beta \in \Phi_{r(F+G)}\backslash\Omega$, there are $R_{\beta r}, S_{\beta r} \in \mathbb{F}^{r}_{\beta I-F-G}$ such that $-\mu R_{\beta r}FG$ and $-\mu S_{\beta r}GF$ are GWDC with generalized set Ω, for any $\mu \in [0,1]$, then*

$$[\sigma_r(F) \cup \sigma_r(G)]\backslash\Omega \subset \sigma_r(F+G)\backslash\Omega.$$

Proof. Let $\beta \in \mathbb{C}\backslash\Omega$. Suppose that there is $R_{\beta r} \in \mathbb{F}^{r}_{\beta I-F-G}$, then we have $(\beta I - F - G)R_{\beta r} = I - K_1$, where $K_1 \in \mathcal{K}(X)$. Thus, we obtain the following:

$$\begin{aligned}(\beta I - F)(\beta I - G) &= \beta(\beta I - F - G) + FG \\ &= \beta(\beta I - F - G) + (\beta I - F - G)R_{\beta r}FG + K_1FG \\ &= (\beta I - F - G)(\beta I + R_{\beta r}FG) + K_1FG.\end{aligned}$$

On the other hand, we have

$$(\beta I - G)(\beta I - F) = (\beta I - F - G)(\beta I + S_{\beta r}GF) + K_1GF.$$

Let $\beta \notin \sigma_r(F+G) \cup \Omega$, it follows that $\beta I - F - G \in \Phi_r(X)$. Let $R_{\beta r}, S_{\beta r} \in \mathbb{F}^r_{\beta I - F - G}$ be such that $-\mu R_{\beta r} FG$ and $-\mu S_{\beta r} GF$ are GWDC with generalized set Ω. From Theorem 3.2 in [5], we infer that $(\beta I + R_{\beta r} FG)$ and $(\beta I + S_{\beta r} GF)$ are Fredholm operators on X, for all $\beta \in \mathbb{C} \backslash \Omega$. Then, using Theorem 2.1, we get $(\beta I - F)(\beta I - G) \in \Phi_r(X)$ and $(\beta I - G)(\beta I - F) \in \Phi_r(X)$. Consequently, we deduce that $\beta I - F \in \Phi_r(X)$ and $\beta I - G \in \Phi_r(X)$, for all $\beta \in \mathbb{C} \backslash \Omega$. □

Theorem 3.3. *Let X be a Banach space and L be a matrix operator defined in Eq.* (2.1) *and L_0 be its closure. Let $\mu \in \rho(A)$ and Ω be a subset of the complex field such that Ω is finite and $0 \in \Omega$. If for $\alpha \in [0,1]$ αA and $\alpha S(\mu)$ are GWDC operators with generalized set Ω, and $\mathbb{T}(\mu)$ is FP, then*

$$\sigma_{ew_i}(L) \backslash \Omega \subset [\sigma_{ew_i}(A) \cup \sigma_{ew_i}(S(\mu))] \backslash \Omega, \quad \textit{where } i \in \{l, r\}.$$

Proof. According to Remark 2.4, we have

$$\begin{aligned} \beta I - L &= \begin{bmatrix} I & 0 \\ F(\mu) & I \end{bmatrix} \begin{bmatrix} \beta I - A & 0 \\ 0 & \beta I - S(\mu) \end{bmatrix} \begin{bmatrix} I & G(\mu) \\ 0 & I \end{bmatrix} - (\beta - \mu)\mathbb{T}(\mu). \\ &:= \mathbb{U}\mathbb{V}(\beta)\mathbb{W} - (\beta - \mu)\mathbb{T}(\mu). \end{aligned} \tag{3.3}$$

Since αA and $\alpha S(\mu)$ are GWDC for $\alpha \in [0,1]$, it follows from Remark 3.1 in [5] and Theorem 3.2 in [5] that $\beta I - A \in \Phi(X)$, $i(\beta I - A) = 0$ and $\beta I - S(\mu) \in \Phi(X)$, $i(\beta I - S(\mu)) = 0$, for all $\beta \in \mathbb{C} \backslash \Omega$.

Now, using Lemma 6.7.1 [8], we obtain $\mathbb{V}(\beta) \in \Phi(X)$ and $i(\mathbb{V}(\beta)) = 0$. Hence we have $\beta I - A \in \Phi(X)$ and $i(\beta I - A) = 0$, which implies that $\beta I - A \in \Phi_l(X)$ and $i(\beta I - A) \leq 0$ (resp. $\beta I - A \in \Phi_r(X)$ and $i(\beta I - A) \geq 0$).

Further, from $\beta I - S(\mu) \in \Phi(X)$ and $i(\beta I - S(\mu)) = 0$, we infer that $\beta I - S(\mu) \in \Phi_l(X)$ and $i(\beta I - S(\mu)) \leq 0$ (resp. $\beta I - S(\mu) \in \Phi_r(X)$ and $i(\beta I - S(\mu)) \geq 0$). Thus, we infer that $\mathbb{V}(\beta) \in \Phi_l(X)$ and $i(\mathbb{V}(\beta)) \leq 0$ (resp. $\mathbb{V}(\beta) \in \Phi_r(X)$ and $i(\mathbb{V}(\beta)) \geq 0$).

Letting $\beta \notin [\sigma_{ew_l}(A) \cup \sigma_{ew_l}(S(\mu))] \cup \Omega$, it follows that $\beta I - A \in \Phi_l(X)$ and $i(\beta I - A) \leq 0$ and $\beta I - S(\mu) \in \Phi_l(X)$ and $i(\beta I - S(\mu)) \leq 0$. Thus, when applying Lemma 6.7.1 [8], we obtain that $\mathbb{V}(\beta) \in \Phi_l(X)$ and $i(\mathbb{V}(\beta)) \leq 0$.

Now, combining the fact that the operators $\mathbb{U}$ and $\mathbb{W}$ are bounded and have bounded inverses and using Eq. (3.3) with the fact that $\mathbb{T}(\mu)$ is FP, we deduce that $\beta I - L \in \Phi_l(X)$ and $i(\beta I - L) \leq 0$. Therefore, we have $\beta \notin \sigma_{ew_l}(L) \cup \Omega$. Hence,

$$\sigma_{ew_l}(L) \backslash \Omega \subset [\sigma_{ew_l}(A) \cup \sigma_{ew_l}(S(\mu))] \backslash \Omega.$$

In the same manner, we show that

$$\sigma_{ew_r}(L) \backslash \Omega \subset [\sigma_{ew_r}(A) \cup \sigma_{ew_r}(S(\mu))] \backslash \Omega.$$

□

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