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ORIGINAL RESEARCH

Generalized weak demicompactness involving measure of non-strict-singularity

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Abstract This paper is devoted to a characterization of the notion of generalized weakly demicompact linear operators acting on a Banach space introduced in [8]. Our results are formulated in terms of measures of non-strict-singularity and strictly singular operators. Theses results are used to discuss the incidence of some perturbation results on the behavior of the closure of unbounded 2×2 block operator matrices.

Keywords Generalized weakly demicompact operator · Fredholm and semi-Fredholm operators · Matrix operator · Essential spectrum · Strictly singular operator

Mathematics Subject Classification 47A53 · 47A10

1 Introduction

The concept of demicompactness appeared in the literature in order to discuss fixed point sets for nonlinear operators acting on Hilbert spaces. It was introduced by W. V. Petryshyn in 1966 as follows: Let X be a Banach space and $T : D \subset X \to X$ a nonlinear operator, then T is said to be demicompact if for every bounded sequence $(x_n)_n$ in D such that $(x_n - Tx_n)_n$ converges in X, then there exists a convergent subsequence of $(x_n)_n$. In the direction of Fredholm theory, W. V. Petryshyn [27] and W. Y. Akashi [3] used this class to provide some results on Fredholm operators. Note that, the class of demicompact operators acting on a Banach space contains the class of compact operators. In 2015, W. Chaker, A. Jeribi and B. Krichen [6] utilized the demicompactness in order to investigate the essential spectra of unbounded closed linear operators. In 2014, B. Krichen [15] introduced the relative demicompactness class with respect to a given closed linear operator, as a generalization of the demicompactness notion, and discussed its incidence on the behavior of some relative essential spectra. In 2018, B. Krichen and D. O'Regan [16] extended this concept to the weak topology setting and defined the so-called weak demicompactness. This definition asserts that an operator $T : D \subset X \to X$, is said to be weakly demicompact if for every bounded sequence $(x_n)_n$ in D such that $(x_n - Tx_n)_n$ converges weakly to an element of X, then there exists a weak convergent subsequence of

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$(x_n)_n$. Recently, in [17] the same authors studied the relationship between the class of weakly demicompact linear operators and measures of weak noncompactness of a linear operator with respect to an axiomatic one. In 2018, F. Ben Brahim, A. Jeribi and B. Krichen [5] provided some sufficient conditions on the inputs of a closable block operator matrix, with domain consisting of vectors which satisfy certain conditions, to ensure the demicompactness of its closure and determined its essential spectra. In 2019, A. Jeribi, B. Krichen and A. Zitouni [11] also used the notion of demicompactness in order to give a new characterization of the essential spectra of a 2×2 block operator matrix. For their study they used the tool of the Frobenius-Schur factorization which was recognized by R. Nagel in [24, 25]. Furthermore, as an application of their theoretical results, the authors studied some systems of differential equations and investigate the essential spectrum of two-groups of transport operators. Moreover, the same authors in [12] established some perturbation results for γ-diagonally dominant operator matrices acting on Banach spaces and illustrated their theoretical results by investigating the essential spectra of operators in Sturm-Liouville problems and in transport equations. In 2019, as generalization of the above notion, I. Ferjani, A. Jeribi and B. Krichen [8] introduced a new concept, related with Riesz operators, the so-called generalized weak demicompactness. Let us recall that a bounded linear operator A is called generalized weakly demicompact, if there exists a finite subset E of $\mathbb{C}$ containing 0 such that

(*i*) For all $\lambda \in \mathbb{C} \backslash E$, $\frac{1}{\lambda}A$ is weakly demicompact operator,
(*ii*) for all $\lambda \in \mathbb{C} \backslash E$, $A - \lambda$ has a finite ascent, and
(*iii*) all $\lambda \in \sigma(A) \backslash E$, are eigenvalues of finite multiplicity and have no accumulation point except possibly points of E.

The set E is called a generalized set of A. Note that, if $E = \{0\}$, A is generalized weakly demicompact if, and only if, $\frac{1}{\lambda}A$ is demicompact for all $\lambda \in \mathbb{C} \backslash \{0\}$ if, and only if, A is Riesz (see Theorem 3.1 [8], Lemma 5.4.6 [10], Theorem 3.1 [6] and Theorem 3.111 [2]). From Remark 3.1 [8], A is also generalized weakly demicompact with generalized set $G \supset E$. This implication is not reversible, indeed if we take $G = \{0, 1\}$ and $A = I$, we get A is generalized weakly demicompact with generalized set G, while $\frac{1}{\lambda}A$ is not demicompact for all $\lambda \in \mathbb{C} \backslash \{0\}$. Inspired by [6, 17], the authors in [8] provided a connection between a generalized weakly demicompact operators and Fredholm and upper semi-Fredholm operators. Moreover, they ensured the generalized weak demicompactness of the closure of a closable block matrix operators.

The aim of this paper is to give a characterization of generalized weak demicompactness by means of measures of non-strict-singularity, extending the well known measures of noncompactness, and strictly singular operators. In particular, we discuss the incidence of some perturbation results on the behavior of the closure of unbounded 2×2 block operator matrices.

In order to state our results, we need first to fix some standard notations and definitions from Fredholm theory. Let X and Y be two Banach spaces and let T be an operator acting from X into Y. We denote by $\mathcal{D}(T) \subset X$ its domain. We denote by $\mathcal{C}(X, Y)$ (resp. $\mathcal{L}(X, Y)$) the set of all closed, densely defined linear operators (resp. the Banach algebra of all bounded linear operators) from X into Y. The subset of $\mathcal{L}(X, Y)$ of all compact operators is denoted by $\mathcal{K}(X, Y)$. For $T \in \mathcal{C}(X, Y)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$, denotes the kernel and the range of T, respectively. The nullity, $\alpha(T)$ is defined as the dimension of $\mathcal{N}(T)$ and the deficiency, $\beta(T)$ is defined as the codimension of $\mathcal{R}(T)$ in Y.

The set of upper semi-Fredholm and lower semi-Fredholm operators from X into Y are defined by

$$\Phi_+(X, Y) := \{T \in \mathcal{C}(X, Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi_-(X, Y) := \{T \in \mathcal{C}(X, Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\}.$$

The set of Fredholm operators from X into Y is defined by

$$\Phi(X, Y) := \Phi_-(X, Y) \cap \Phi_+(X, Y).$$

If $T \in \Phi(X, Y)$, the number $i(T) := \alpha(T) - \beta(T)$ is called the index of T. The set Φ_T is defined by

$$\Phi_T := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \in \Phi(X, Y)\}.$$

Now, a bounded operator F is called an upper semi-Fredholm perturbation if $U + F \in \Phi_+(X, Y)$ whenever $U \in \Phi_+(X, Y)$. We denote by $\mathcal{F}_+(X, Y)$ the set of upper semi-Fredholm perturbation.

Note that if $X = Y$, then $\mathcal{L}(X, Y)$, $\mathcal{C}(X, Y)$, $\mathcal{K}(X, Y)$, $\Phi(X, Y)$, $\Phi_+(X, Y)$ and $\Phi_-(X, Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$, respectively. If $T \in \mathcal{C}(X)$, we denote by $\rho(T)$ the resolvent set of T and by $\sigma(T)$ the spectrum of T.

The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [13] as a generalization of the notion of compact operators as follows: An operator $S \in \mathcal{L}(X, Y)$ is said to be strictly singular if the restriction of S to any infinite-dimensional subspace of X is not an homeomorphism. We denote by $\mathcal{S}(X, Y)$ the set of strictly singular operators from X to Y. Note that $\mathcal{S}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$. In general, strictly singular operators are not compact and if $X = Y$, $\mathcal{S}(X) = \mathcal{S}(X, X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. For a detailed study of the properties of strictly singular operators we refer to [9, 20, 26, 33].

To provide our main results, we start by recalling the definition of Hausdorff measure of noncompactness cited in [29]. The notion of measure of noncompactness, introduced by Kuratwskii [14] in 1930, turned out to be a useful tool in some topological problems and in operator theory as in [1].

Definition 1.1 Let X and Y be two Banach spaces. The Hausdorff measure of an operator $A \in \mathcal{L}(X, Y)$ is defined by

$$q(A) = q[A(B_X)] := \inf\{r > 0, A(B_X) \text{ can be covred by finite set of open ball of radius } r\},$$

where B_X denotes the closed unit ball in X, that is, the set of all $x \in X$ satisfying $\|x\| \leq 1$. $\diamond$

It was proved in [18] that

$$\begin{aligned} &q(A) \leq \|A\|, \\ &q(A) = 0 \text{ if, and only if, } A \in \mathcal{K}(X, Y), \text{ and} \\ &q(A + K) = q(A), \text{ for all } K \in \mathcal{K}(X, Y). \end{aligned}$$

After that, this concept was generalized to a new concept of measuring called measure of non-strict-singularity.

Definition 1.2 Let X and Y be two Banach spaces and $A \in \mathcal{L}(X, Y)$. We consider

$$g_M = \inf_{N \subset M} q(A_{|N}) \text{ and } g(A) = \sup_{M \subset X} g_M(A), \tag{1.1}$$

where M, N are two infinite dimensional subspaces of X, and $A_{|N}$ denotes the restriction of A to the subspace N. $\diamond$

This notion of the measure g enjoys many interesting property established by V. R. Rakočević in [28] for future use.

Proposition 1.1 [28] *Let X and Y be two Banach spaces. For $A \in \mathcal{L}(X, Y)$,*

(i) $A \in \mathcal{S}(X, Y)$ *if, and only if,* $g(A) = 0$.
(ii) $A \in \mathcal{S}(X, Y)$ *if, and only if,* $g(A + T) = g(T)$ *for all* $T \in \mathcal{L}(X, Y)$.
(iii) *If Z is a Banach space and $B \in \mathcal{L}(Y, Z)$, then $g(BA) \leq g(B)g(A)$.*

$\diamond$

Motivated by this measure g, our interest concentrates to gather a new characterization of generalized weakly demicompact operators. More specifically, inspired by [17, 21], we prove that for $A \in \mathcal{L}(X)$ and P be a non zero complex polynomial with zeros $z_1, \cdots, z_n$. If $\lim_{n \to 0} g(P(A)^n)^{\frac{1}{n}} = 0$, then μA is generalized weakly demicompact for all $\mu \in [0, 1]$ with a generalized set $E = \{0, z_1, \cdots, z_n\}$.

Moreover, we find a class of perturbation for the generalized weakly demicompact operators, more precisely, let us consider the following set

$$\mathcal{I}_n(X) = \{K \in \mathcal{L}(X) \text{ satisfying } g((BK)^n) < 1 \text{ for all } B \in \mathcal{L}(X)\},$$

we show that if A is generalized weakly demicompact, then for all $K \in \mathcal{I}_n(X)$, we have $A + K$ is generalized weakly demicompact.

Various notions of essential spectra have been defined in the literature. In this work, we are concerned with the following essential spectra:

$$\sigma_{e_1}(T) = \{\alpha \in \mathbb{C} \text{ such that } \alpha - T \notin \Phi_+(X)\} := \mathbb{C}\backslash\Phi_{+T},$$
$$\sigma_{e_4}(T) = \{\alpha \in \mathbb{C} \text{ such that } \alpha - T \notin \Phi(X)\} := \mathbb{C}\backslash\Phi_T,$$
$$\sigma_{e_5}(T) = \mathbb{C}\backslash\rho_5(T), \text{ where}$$
$$\rho_5(T) = \{\alpha \in \mathbb{C} \text{ such that } \alpha \in \Phi_T \text{ and } i(\alpha - T) = 0\}.$$

The second aim of this work is to derive the stability of the essential spectrum of the sum of two bounded linear operators by means of the essential spectrum of each of the two operators. Finally, in order to extend some results founded in [21, 22], we apply the results described above to study the invariance of the essential spectra of a 2×2 block operator matrices by using a measure product of non-strict-singularity.

The structure of this paper is as follows: In Section 2, we establish the relationship between generalized weakly demicompact and strictly singular operators. Moreover, a characterization by means of measure of non-strict-singularity is given. In Section 3, the obtained results are used to discuss the invariance of the essential spectrum. In Section 4, we provide some sufficient conditions on the inputs of a closable block operator matrix to ensure the generalized weak demicompactness of its closure.

2 Characterization of generalized weakly demicompact operators

In this section, we will show the connection between the generalized weakly demicompact and the strictly singular operators. Moreover, we give a characterization of this class by means of measure of non-strict-singularity.

Theorem 2.1 *Let X be a Banach space, $A \in \mathcal{L}(X)$ and P be a non zero complex polynomial with zeros $z_1, \cdots, z_n$. If $\lim\limits_{n\to 0} g(P(A)^n)^{\frac{1}{n}} = 0$, then μA is generalized weakly demicompact for all $\mu \in [0,1]$ with a generalized set $E = \{0, z_1, \cdots, z_n\}$.* $\diamond$

Proof Let $\mu \in [0,1]$. Suppose that μA is not generalized weakly demicompact with a generalized set $E = \{0, z_1, \cdots, z_n\}$, then from Theorem 3.5 in [8], it implies that $\mu P(A)$ is not generalized weakly demicompact with a generalized set $E' = \{0\}$.

Now, applying Theorem 3.1 in [8], there exists $\lambda \in \mathbb{C}\backslash\{0\}$ such that $\lambda I - \mu P(A) \notin \Phi_+(X)$. From Lemma 4.3 in [18], it follows that there exists an infinite dimensional subspace M of X such that the restriction $[\lambda I - \mu P(A)]_{|M}$ of $\lambda I - \mu P(A)$ to M is in $\mathcal{K}(M, X)$.

Consequently, we have

$$g([\lambda I - \mu P(A)]_{|M}) = 0.$$

It follows that

$$g(\lambda I_{|M}) = g(\mu P(A)_{|M}) \leq g(\mu P(A)).$$

On the other hand, since $\lim\limits_{n\to 0} g(P(A)^n)^{\frac{1}{n}} = 0$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$,

$$g([P(A)]^{n_0})^{\frac{1}{n_0}} < |\lambda|.$$

Also, we can write, for $n \geq n_0$,

$$[\lambda^{n_0} I - (\mu P(A))^{n_0}]_{|M} = \sum_{j=0}^{n_0-1} \lambda^j [\mu P(A)]_{|M}^{n_0-1-j} [\lambda I - \mu P(A)]_{|M}. \tag{2.1}$$

Combining the fact that $g([\lambda I - \mu P(A)]_{|M}) = 0$ and Equation (2.1), we infer that

$$g\left([\lambda^{n_0} I - [\mu P(A)]^{n_0}]_{|M}\right) = 0.$$

We deduce that

$$g\left((\lambda^{n_0} I)_{|M}\right) = g\left([\mu P(A)]_{|M}^{n_0}\right) \leq g([\mu P(A)]^{n_0}).$$

Which implies that

$$|\lambda|^{n_0} g(I) \leq |\mu|^{n_0} g([P(A)]^{n_0}) \leq g([P(A)]^{n_0}).$$

Then, we get

$$|\lambda| \leq g([P(A)]^{n_0})^{\frac{1}{n_0}} < |\lambda|.$$

But this is impossible. Hence, we must have $\lambda I - \mu P(A) \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$ and all $\mu \in [0,1]$. Thus, μA is generalized weakly demicompact for all $\mu \in [0,1]$, with a generalized set $E = \{0, z_1, \cdots, z_n\}$. □

As immediate consequence of Theorem 2.1 is the following corollary.

Corollary 2.1 *Let A be a bounded linear operator acting on a Banach space X. Suppose that there exists a complex polynomial P such that $g(P(A)) < 1$. Then, μA is generalized weakly demicompact for all $\mu \in [0,1]$.* ◇

Theorem 2.2 *Let X be a Banach space and $A \in \mathcal{L}(X)$. If A is a strictly singular operator, then A is a generalized weakly demicompact operator.* ◇

Proof Suppose that A is a strictly singular operator. Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a complex polynomial with zeros $z_1, \cdots, z_n$ and g be a measure of non-strict-singularity on X, then we have

$$\begin{aligned} g(P(A)) &= g\left(\sum_{k=0}^{n} a_k A^k\right) \\ &\leq \sum_{k=0}^{n} |a_k| g(A^k) \\ &\leq \sum_{k=0}^{n} |a_k| (g(A))^k = 0. \end{aligned}$$

This implies that $g(P(A)) = 0$. Hence, from Proposition 1.1, it follows that $P(A)$ is strictly singular operator.
Now, Let Z be a Banach space, $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}\backslash\{0\}$, then

$$\begin{aligned} g(T) &= g\left(\left(I - \frac{1}{\lambda}P(A)\right)T + \frac{1}{\lambda}P(A)T\right) \\ &\leq \frac{1}{|\lambda|} g((\lambda I - P(A))T) + \frac{1}{|\lambda|} g(P(A)T) \\ &\leq \frac{1}{|\lambda|} g((\lambda I - P(A))T) + \frac{1}{|\lambda|} g(P(A)) g(T). \end{aligned}$$

Taking into account the fact that $P(A)$ is a strictly singular operator, it follows from Proposition 1.1, that $g(P(A)) = 0$. Thus, we obtain

$$g(T) \leq \frac{1}{|\lambda|} g((\lambda I - P(A))T).$$

Therefore, when applying Theorem 3.5 [31], we get $\lambda I - P(A) \in \Phi_+(X)$, for all $\lambda \in \mathbb{C}\backslash\{0\}$. Hence, from Theorem 3.1 in [8], $P(A)$ is a generalized weakly demicompact operator with a generalized set $E = \{0\}$. Consequently, in view of Theorem 3.5 in [8], A is a generalized weakly demicompact operator with a generalized set $E' = \{0, z_1, \cdots, z_n\}$. □

Now, For $n \in \mathbb{N}$, let us consider the following set

$$\mathcal{I}_n(X) = \{K \in \mathcal{L}(X) \text{ satisfying } g((BK)^n) < 1 \text{ for all } B \in \mathcal{L}(X)\}.$$

We have the following inclusion

$$\mathcal{S}(X) \subset \mathcal{I}_n(X).$$

Remark 2.1 (*i*) If $K \in \mathcal{I}_n(X)$ and $A \in \mathcal{L}(X)$, then $AK \in \mathcal{I}_n(X)$.
(*ii*) If $K \in \mathcal{I}_n(X)$ and $S \in \mathcal{S}(X)$, then $K + S \in \mathcal{I}_n(X)$. ◇

Remark 2.2 Let X be a Banach space and $A \in \mathcal{L}(X)$. If $A \in \mathcal{I}_n(X)$, then A is a generalized weakly demicompact operator. Indeed, let $A \in \mathcal{I}_n(X)$, then we obtain that $g((BA)^n) < 1$ for all $B \in \mathcal{L}(X)$. Thus, we can take $B = I$, so we get $g(A^n) < 1$. Therefore, from Corollary 2.1, A is a generalized weakly demicompact operator. ⋄

Theorem 2.3 *Let X be a Banach space and $A \in \mathcal{L}(X)$. If A is a generalized weakly demicompact operator, then for all $K \in \mathcal{I}_n(X)$, we have $A + K$ is a generalized weakly demicompact operator.* ⋄

Proof Suppose that A is a generalized weakly demicompact, then from Theorem 3.1 [8], we have $\lambda I - A \in \Phi_+(X)$, for all $\lambda \in \mathbb{C}\backslash E$. Thus, there exist $F \in \mathcal{K}(X)$ and $A_0 \in \mathcal{L}(X)$ such that

$$A_0(\lambda I - A) = I - F.$$

Now, let us take $K \in \mathcal{I}_n(X)$, thus we have

$$\begin{aligned} A_0(\lambda I - (A + K)) &= A_0(\lambda I - A - K) \\ &= A_0(\lambda I - A) - A_0 K \\ &= I - F - A_0 K. \end{aligned}$$

Since $K \in \mathcal{I}_n(X)$, we have $g((A_0K)^n) < 1$, so it follows from Theorem 2.1, that A_0K is a generalized weakly demicompact operator with a generalized set $E' = \{0\}$. Thus, from Theorem 3.1 in [8], we get $\gamma I - A_0K \in \Phi_+(X)$ for all $\gamma \in \mathbb{C}\backslash\{0\}$, in particular we have $I - A_0K \in \Phi_+(X)$. Therefore, taking into account the fact that $F \in \mathcal{K}(X)$, we deduce that $A_0(\lambda I - (A + K)) \in \Phi_+(X)$. Hence, in view of Theorem 6 [23], $\lambda I - (A + K) \in \Phi_+(X)$, for all $\lambda \in \mathbb{C}\backslash E$. Now, when applying Theorem 3.1 in [8], it yields that $A + K$ is a generalized weakly demicompact operator. □

Now, we will give some perturbation results and the relationship between the essential spectrum of the sum of two linear operators and the essential spectrum of each one.

Theorem 2.4 *Let X be a Banach space, $T \in \mathcal{L}(X)$ and $A \in \mathcal{L}(X)$. Assume that there exists a finite subset E of $\mathbb{C}$ containing 0 such that one of the following assertions holds:*

(i) *For every $\lambda \in \Phi_{+(T+A)}\backslash E$, there exist $F_{\lambda l}$ and $H_{\lambda l}$ a left Fredholm inverses operators of $(\lambda I - T - A)$ such that $g(-TAF_{\lambda l}) < 1$ and $g(-ATH_{\lambda l}) < 1$.*
(ii) *For every $\lambda \in \Phi_{+(T+A)}\backslash E$, there exist $F_{\lambda r}$ and $H_{\lambda r}$ a right Fredholm inverses operators of $(\lambda I - T - A)$ such that $g(-F_{\lambda r}TA) < 1$ and $g(-H_{\lambda r}AT) < 1$.*

Then,

$$[\sigma_{e_1}(T) \cup \sigma_{e_1}(A)]\backslash E \subset [\sigma_{e_1}(T + A)]\backslash E.$$

⋄

Proof (*i*) Let $\lambda \in \mathbb{C}\backslash E$, if there exists $F_{\lambda l}$ a left Fredholm inverse operators of $(\lambda I - T - A)$ then $F_{\lambda l}(\lambda I - T - A) = I - K$ where $K \in \mathcal{K}(X)$, then we have

$$\begin{aligned} (\lambda I - T)(\lambda I - A) &= \lambda(\lambda I - T - A) + TA \\ &= \lambda(\lambda I - T - A) + TAF_{\lambda l}(\lambda I - T - A) + TAK. \end{aligned}$$

We conclude

$$(\lambda I - T)(\lambda I - A) = (\lambda I + TAF_{\lambda l})(\lambda I - T - A) + TAK. \tag{2.2}$$

In the same manner, we can write

$$(\lambda I - A)(\lambda I - T) = (\lambda I + ATH_{\lambda l})(\lambda I - T - A) + ATK. \tag{2.3}$$

Let $\lambda \notin \sigma_{e_1}(T + A) \cup E$, then $\lambda I - T - A \in \Phi_+(X)$ and set $F_{\lambda l}$ and $H_{\lambda l}$ a left Fredholm inverses operators of $(\lambda I - T - A)$ such that $g(-TAF_{\lambda l}) < 1$ and $g(-ATH_{\lambda l}) < 1$. Now, applying Theorem 2.1 and Theorem 3.1 [8], we get $(\lambda I + TAF_{\lambda l})$ and $(\lambda I + ATH_{\lambda l})$ are upper semi-Fredholm operators on X, for all $\lambda \in \mathbb{C}\backslash E$. Thus, when using Theorem 5.26 in [30], Equation (2.2) and Equation (2.3), we prove that $(\lambda I - T)(\lambda I - A)$ and $(\lambda I - A)(\lambda I - T)$ are upper semi-Fredholm operators both. Now, from theorem 6 in [23], we complete the proof for the first case.

(ii) If there exist $F_{\lambda r}$ and $H_{\lambda r}$ a right Fredholm inverses operators of $(\lambda I - T - A)$, then we can write

$$(\lambda I - T)(\lambda I - A) = (\lambda I - T - A)(\lambda I + F_{\lambda r}TA) + K_1 TA. \tag{2.4}$$

and

$$(\lambda I - A)(\lambda I - T) = (\lambda I - T - A)(\lambda I + H_{\lambda r}AT) + K_1 AT, \tag{2.5}$$

where $K_1 \in \mathcal{K}(X)$.

The same arguments have been used, it is sufficient to replace Equation (2.2) by Equation (2.4) or Equation (2.3) by Equation (2.5). $\square$

Theorem 2.5 *Let X be a Banach space, $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(X)$ E and let be a finite subset of $\mathbb{C}$ containing 0. If for every $\lambda \in \Phi_{+(A+B)}\backslash E$, there exists $F_{\lambda l}$ (resp. $F_{\lambda r}$) a left (resp. right) Fredholm inverse operators of $(\lambda I - A - B)$ such that $-\mu ABF_{\lambda l}$ is a strictly singular operator for any $\mu \in [0, 1]$, then*

$$\sigma_{e_4}(A + B)\backslash E \subset [\sigma_{e_4}(A) \cup \sigma_{e_4}(B)]\backslash E.$$

Moreover, if there exists $H_{\lambda l}$ a left Fredholm inverse operators of $(\lambda I - A - B)$ such that $-\mu BAH_{\lambda l}$ is strictly singular operator for any $\mu \in [0, 1]$, then

$$\sigma_{e_4}(A + B)\backslash E = [\sigma_{e_4}(A) \cup \sigma_{e_4}(B)]\backslash E.$$

$\diamond$

Proof Let $\lambda \in \mathbb{C}\backslash E$. If there exists $F_{\lambda l}$ a left Fredholm inverse operators of $(\lambda I - A - B)$, then there exists $K \in \mathcal{K}(X)$ such that $F_{\lambda l}(\lambda I - A - B) = I - K$. Thus, we have

$$\begin{aligned}(\lambda I - A)(\lambda I - B) &= \lambda(\lambda I - A - B) + AB \\ &= \lambda(\lambda I - A - B) + ABF_{\lambda l}(\lambda I - A - B) + ABK \\ &= (\lambda I + ABF_{\lambda l})(\lambda I - A - B) + ABK,\end{aligned}$$

Let $\lambda \notin [\sigma_{e_4}(A) \cup \sigma_{e_4}(B)] \cup E$, then $\lambda I - A \in \Phi(X)$ and $\lambda I - B \in \Phi(X)$. It follows from Theorem 6 in [23] that $(\lambda I - A)(\lambda I - B) \in \Phi(X)$. As $-\mu ABF_{\lambda l}$ is strictly singular, it follows from Theorem 2.2 and Theorem 3.2 in [8] that $(\lambda I + ABF_{\lambda l})$ is a Fredholm operator. Now, using the fact that ABK is compact, we conclude together with Theorem 7.12 in [30] that $\lambda I - A - B \in \Phi(X)$. Hence, $\lambda \notin \sigma_{e_4}(A + B)$.

Conversely, if $\lambda \notin \sigma_{e_4}(A + B) \cup E$, then arguing the operators as above, we deduce that $(\lambda I - A)(\lambda I - B) \in \Phi(X)$ and $(\lambda I - B)(\lambda I - A) \in \Phi(X)$. According to Theorem 5.14 in [30], we conclude that $\lambda I - A \in \Phi(X)$ and $\lambda I - B \in \Phi(X)$.

Thus, it yields that $\lambda \notin [\sigma_{e_4}(A) \cup \sigma_{e_4}(B)]\backslash E$. $\square$

Now, we will show the applicability of Theorem 2.1 to a bounded linear operators arising to the differential equations.

Example 2.1 Let $C([0,\ w])$ be the space of continuous w-periodic functions $x : \mathbb{R} \to \mathbb{R}^n$ and $C^3([0, w])$ the space of 3-times continuously differentiable w-periodic functions $x : \mathbb{R} \to \mathbb{R}^n$. $C([0,\ w])$ equipped with the maximum norm $\| \cdot \|_\infty$ and $C^3([0, w])$ with the norm given by $\|u\|^1_\infty = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$ for $u \in C^3([0, w])$ are Banach spaces. Let us consider the following differential equation:

$$x'''(t) = a(t)x'''(t - h_1) + b(t)x''(t - h_2) + c(t)x'(t - h_3) + d(t)x(t - h_4).$$

Here, a, b, c and d are continuous w-periodic functions such that $|a(t)| < k$, $(k < 1)$, where $k < \frac{1}{w}$ if $w > 2$ or $k < \frac{1}{2}$ if $w \leq 2$ and $x \in C^3([0, w])$ is an unknown function.

Let us now consider the complex polynomial $P(X) = \frac{1}{3}X^3(X^3 - 1)$ and the operator $T = c^{D^{(\frac{1}{3})}}$, where $c^{D^{(\frac{1}{3})}}$ is the Caputo derivative of fractional order $\frac{1}{3}$.

T is a bounded operator. Indeed, first write

$$Tx(t) = \frac{3}{5\Gamma(\frac{5}{3})} \int_0^t (t - \tau)^{\frac{5}{3}} x'''(\tau)d\tau,$$

and let us consider the set $U := \{x \in C^3[0, w]$ such that $\|x'''\|_\infty \leq k\}$.

Now, we have $T(U) := \{Tx,\ x \in U\}$. Note that, for $z \in T(U)$ and $t \in [0, w]$, we have the following estimate:

$$|z(t)| \le \frac{9}{40\Gamma(\frac{5}{3})}\|x'''\|_\infty w^{\frac{8}{3}},$$

which is the required boundedness.

Now, when applying Theorem 3.4 [19] and Definition 2.2 [7], we obtain

$$P(T)(x) = \frac{1}{3}T^3(T^3 - I)(x) = \frac{1}{3}\left[{}^{c}D^{(\frac{1}{3})}\right]^3 \left(\left[{}^{c}D^{(\frac{1}{3})}\right]^3 - I\right)x(t) = \frac{1}{3}[x''(t) - x'(t)].$$

Clearly, $P(T)$ is a bounded operator and $\|P(T)\| \le \frac{2}{3} < 1$.

Now, from Proposition 2.1 [21], we have

$$g(P(T)) \le q(P(T)),$$

and taking into account that

$$q(P(T)) \le \|P(T)\|.$$

Hence we get

$$g(P(T)) < 1.$$

Thus, from Theorem 2.1, we deduce that μT is a generalized weakly demicompact operator with generalized set $E = \{0, 1, e^{\frac{2i\pi}{3}}, e^{\frac{-2i\pi}{3}}\}$, for $\mu \in [0, 1]$.

3 Generalized weak demicompactness results for a matrix operator

In this section we will use some conditions introduced in [32], imposed on the entries of the operator L_0 to ensure its closedness. Having obtained the closure of the operator L_0, we will investigate after its essential spectra. For this we consider, in the product space $X \times X$, the following matrix operator

$$L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.1}$$

where the operator A acts on X and has domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space X, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp., $\mathcal{D}(C)$) and acts on X. Now, let us consider the following assumptions [32].

- $(H1)$ A is closed, densely defined linear operator on X with nonempty resolvent set $\rho(A)$.
- $(H2)$ The operator B is a densely defined linear operator on X and for some (hence all) $\mu \in \rho(A)$, the operator $(A - \mu)^{-1}B$ is closable. (In particular, if B is closable, then $(A - \mu)^{-1}B$ is closable).
- $(H3)$ The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence all) $\mu \in \rho(A)$, the operator $C(A - \mu)^{-1}$ is bounded. (In particular, if C is closable, then $C(A - \mu)^{-1}$ is bounded).
- $(H4)$ $\mathcal{D}(B) \subset \mathcal{D}(D)$.
- $(H5)$ For some (hence all) $\mu \in \rho(A)$, the operator $D - C(A - \mu)^{-1}B$ is closable. We will denote by $S(\mu)$ its closure.

Remark 3.1 (i) Under the assumptions $(H1)$ and $(H2)$, we infer that for each $\mu \in \rho(A)$ the operator $G(\mu) := \overline{(A - \mu)^{-1}B}$ is bounded on X.

(ii) From the assumption $(H3)$, it follows that the operator: $F(\mu) := C(A - \mu)^{-1}$ is bounded on X.$\diamond$

We recall the following result which describes the closure of L_0.

Theorem 3.1 [4] Let conditions $(H1)$-$(H3)$ be satisfied and the lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ be dense in X. Then the operator L_0 is closable and the closure L of L_0 is given by:

$$L = \mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - A & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}. \tag{3.2}$$

$\diamond$

Or, spelled out,

$$\begin{aligned} L : \mathcal{D}(L) \subset (X \times X) &\longrightarrow X \times X \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(x + G(\mu)y) - \mu G(\mu)y \\ C(x + G(\mu)y) - S(\mu)y \end{pmatrix}, \end{aligned}$$

with $\mathcal{D}(L) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times X \text{ such that } x + G(\mu)y \in \mathcal{D}(A) \text{ and } y \in \mathcal{D}(S(\mu)) \right\}$.

Note that the description of the operator L does not depend on the choice of the point $\mu \in \rho(A)$.

Remark 3.2 Let $\lambda \in \mathbb{C}$. It follows from Equation (3.2) that

$$\begin{aligned} \lambda I - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda I - A & 0 \\ 0 & \lambda I - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \mu)M(\mu) \\ &:= UV(\lambda)W - (\lambda - \mu)M(\mu), \end{aligned}$$

where $M(\mu) = \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}$. ◇

Theorem 3.2 *Let X be a Banach space. Assume that the operator L_0 defined in Equation (3.1) satisfies $(H_1)-(H_5)$. Let $\mu \in \rho(A)$ and $\lambda \in \mathbb{C}$. If A and $S(\mu)$ are strictly singular operators, then L is a generalized weakly demicompact operator.* ◇

Proof According to Theorem 3.1, we have

$$\begin{aligned} \lambda I - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda I - A & 0 \\ 0 & \lambda I - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}. \\ &:= UV(\lambda)W \end{aligned} \tag{3.3}$$

Since A, $S(\mu)$ are strictly singular operators, it follows from Theorem 2.2, that A and $S(\mu)$ are generalized weakly demicompact operators. Now, when applying Theorem 3.1 in [8], there exist two finite subsets E_1 and E_2 of $\mathbb{C}$ containing 0 such that $\lambda I - A \in \Phi_+(X)$, for all $\lambda \in \mathbb{C} \backslash E_1$ and $\lambda I - S(\mu) \in \Phi_+(X)$, for all $\lambda \in \mathbb{C} \backslash E_2$. From Remark 3.1 [8], we get $\lambda I - A \in \Phi_+(X)$ and $\lambda I - S(\mu) \in \Phi_+(X)$, for all $\lambda \in \mathbb{C} \backslash E$, where $E = E_1 \cup E_2$. So, from Lemma 6.6.1 [10], we obtain $V(\lambda) \in \Phi_+(X \times X)$, for all $\lambda \in \mathbb{C} \backslash E$.

Taking into account that the operators U and W are invertible with a bounded inverses, we deduce that $UV(\lambda)W \in \Phi_+(X \times X)$, for all $\lambda \in \mathbb{C} \backslash E$. From Equation (3.3), we have $\lambda I - L$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C} \backslash E$. Hence, we infer from Theorem 3.1 [8] that L is a generalized weakly demicompact operator with a generalized set E. □

Theorem 3.3 *Let X be a Banach space. Assume that the operator L_0 defined in Equation (3.1)* satisfies *$(H_1)-(H_5)$. Let $\mu \in \rho(A)$, $\lambda \in \mathbb{C}$ and $\alpha \in [0, 1]$. Assume that $g(A) < 1$ and $g(S(\mu)) < 1$ and $(\lambda - \alpha\mu)M(\mu) \in \mathcal{I}_n(X)$, then αL is a generalized weakly demicompact operator, for all $\alpha \in [0, 1]$.* ◇

Proof In view of Remark 3.2, we have

$$\begin{aligned} \lambda I - \alpha L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda I - \alpha A & 0 \\ 0 & \lambda I - \alpha S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \alpha\mu)M(\mu) \\ &:= UV_\alpha(\lambda)W - (\lambda - \alpha\mu)M(\mu), \end{aligned} \tag{3.4}$$

Since $g(A) < 1$ and $g(S(\mu)) < 1$, it follows from Corollary 2.1, that αA and $\alpha S(\mu)$ are generalized weakly demicompact operators for all $\alpha \in [0, 1]$. Now, by applying Theorem 3.1 [8] and from Remark 3.1 [8], there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda I - \alpha A \in \Phi_+(X)$ and $\lambda I - \alpha S(\mu) \in \Phi_+(X)$, for all $\lambda \in \mathbb{C} \backslash E$. So, from Lemma 6.6.1 [10], we obtain $V_\alpha(\lambda) \in \Phi_+(X \times X)$.

Taking into account the fact that U and W are invertible with a bounded inverses, we deduce that $UV_\alpha(\lambda)W \in \Phi_+(X \times X)$.

From Equation (3.4), we have

$$\lambda I - \alpha L = UV_\alpha(\lambda)W + (\lambda - \alpha\mu)M(\mu).$$

From the fact that $UV_\alpha(\lambda)W$ is an upper semi-Fredholm operator and the fact that $(\lambda - \alpha\mu)M(\mu) \in \mathcal{I}_n(X)$, we infer from Theorem 2.3 that, αL is a generalized weakly demicompact operator, for all $\alpha \in [0, 1]$. □

Theorem 3.4 *Let X be a Banach space. Assume that the operator L_0 defined in Equation (3.1) satisfies $(H_1)-(H_5)$. Let $\mu \in \rho(A)$, E be a finite subset of $\mathbb{C}$ containing 0 and $\alpha \in [0,1]$. Suppose that $F(\mu)$ and $G(\mu)$ are strictly singular operators, then*

(i) *If A and $S(\mu)$ are strictly singular operators, then*

$$\sigma_{e_1}(L)\backslash E = [\sigma_{e_1}(A) \cup \sigma_{e_1}(S(\mu))]\backslash E.$$

(ii) *If $g(A) < 1$ and $g(S(\mu)) < 1$, then*

$$\sigma_{e_i}(L)\backslash E = [\sigma_{e_i}(A) \cup \sigma_{e_i}(S(\mu))]\backslash E, \text{ where } i \in \{4,5\}.$$

Proof (i) Let $\mu \in \rho(A)$. Since A and $S(\mu)$ are strictly singular operators, it follows from Theorem 3.2 that the operator L is a generalized weakly demicompact with a generalized set E. Hence, in view of Theorem 3.1 [8], we get $\lambda I - L \in \Phi_+(X \times X)$, for all $\lambda \in \mathbb{C}\backslash E$.

Now, using the fact that $F(\mu)$ and $G(\mu)$ are strictly singular operators, we infer from Proposition 1.1 that $M(\mu)$ is also a strictly singular operator. Thus, from the following inclusion $\mathcal{S}(X) \subset \mathcal{F}_+(X)$, we obtain that $(\lambda - \mu)M(\mu) \in \mathcal{F}_+(X \times X)$. So, from this fact, it yields that $\lambda I - L \in \Phi_+(X \times X)$ if, and only if, the operator $UV(\lambda)W$ is such too. Now, taking into account the fact that U and W are invertible with bounded inverses, we deduce that $\lambda I - L \in \Phi_+(X \times X)$ if, and only if, $V(\lambda) \in \Phi_+(X \times X)$ if, and only if, $\lambda I - A \in \Phi_+(X)$ and $\lambda I - S(\mu) \in \Phi_+(X)$, for all $\lambda \in \mathbb{C}\backslash E$. Thus,

$$\sigma_{e_1}(L)\backslash E = [\sigma_{e_1}(A) \cup \sigma_{e_1}(S(\mu))]\backslash E.$$

(ii) We recall the following equation

$$\begin{aligned}\lambda I - \alpha L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}\begin{pmatrix} \lambda I - \alpha A & 0 \\ 0 & \lambda I - \alpha S(\mu) \end{pmatrix}\begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \alpha\mu)M(\mu) \\ &:= UV_\alpha(\lambda)W - (\lambda - \alpha\mu)M(\mu),\end{aligned}$$

Since $F(\mu)$ and $G(\mu)$ are strictly singular operators, it follows from Proposition 1.1 that $M(\mu)$ is also a strictly singular operator. Now, when we use the following inclusion $\mathcal{S}(X) \subset \mathcal{I}_n(X)$, we infer that $(\lambda - \alpha\mu)M(\mu) \in \mathcal{I}_n(X)$. Thus, it follows from the two inequalities $g(A) < 1$ and $g(S(\mu)) < 1$, and Theorem 3.3 that αL is a generalized weakly demicompact operator for all $\alpha \in [0,1]$ with a generalized set E.

Hence, in view of Theorem 3.2 [8], we deduce that $\lambda I - L \in \Phi(X \times X)$ and $i(\lambda I - L) = 0$.

Now, a similar reasoning as (i) allows us to conclude that

$$\sigma_{e_i}(L)\backslash E = [\sigma_{e_i}(A) \cup \sigma_{e_i}(S(\mu))]\backslash E, \text{ where } i \in \{4,5\}.$$

□

Note that condition $(\lambda - \alpha\mu)M(\mu) \in \mathcal{I}_n(X)$ in Theorem 3.3 is not easy to check.

However, the following lemma provide a measure product of non-strict-singularity which can be used under some sufficient conditions on $F(\mu)$ and $G(\mu)$ of the Frobenius-Schur decomposition in order to verify it.

Lemma 3.1 *Suppose g is a measure of non-strict-singularity in Banach space X. Moreover, assume that the function $\Delta : [0, +\infty[^4 \longrightarrow [0, +\infty[$ is convex and $\Delta(x_1, x_2, x_3, x_4) = 0$ if, and only if, $x_i = 0$ for $i = 1,2,3,4$.*

For all bounded operator

$$\mathcal{T} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

on $X \times X$, we consider

$$\mu^*(\mathcal{T}) = \Delta(g(T_1), g(T_2), g(T_3), g(T_4)).$$

Then μ^* is a measure of non-strict-singularity on $X \times X$.$\diamond$

Proof In the first step, we will check that μ^* is a semi-norm.

(i) Let $\mathcal{T} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ and $\mathcal{S} = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$.

Then, we have

$$\begin{aligned}\mu^*(\mathcal{T}+\mathcal{S}) &= \Delta(g(T_1+S_1), g(T_2+S_2), g(T_3+S_3), g(T_4+S_4))\\ &\leq \Delta(g(T_1)+g(S_1), g(T_2)+g(S_2), g(T_3)+g(S_3), g(T_4)+g(S_4))\\ &\leq \Delta(g(T_1), g(T_2), g(T_3), g(T_4)) + \Delta(g(S_1), g(S_2), g(S_3), g(S_4)).\end{aligned}$$

Hence, we conclude that

$$\mu^*(\mathcal{T}+\mathcal{S}) \leq \mu^*(\mathcal{T}) + \mu^*(\mathcal{S}).$$

(ii) Let $\lambda \in \mathbb{C}^*$ and $\mathcal{T} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$.

$$\begin{aligned}\mu^*(\lambda\mathcal{T}) &= \Delta(g(\lambda T_1), g(\lambda T_2), g(\lambda T_3), g(\lambda T_4))\\ &= \Delta(|\lambda| g(T_1), |\lambda| g(T_2), |\lambda| g(T_3), |\lambda| g(T_4))\\ &\leq |\lambda| \Delta(g(T_1), g(T_2), g(T_3), g(T_4)) \leq |\lambda| \mu^*(\mathcal{T}).\end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}\mu^*(\mathcal{T}) &= \Delta(g(T_1), g(T_2), g(T_3), g(T_4))\\ &= \Delta(\frac{1}{|\lambda|}|\lambda|(g(T_1), g(T_2), g(T_3), g(T_4)))\\ &\leq \frac{1}{|\lambda|}\Delta(|\lambda|(g(T_1), g(T_2), g(T_3), g(T_4))).\end{aligned}$$

Which implies that

$$|\lambda|\mu^*(\mathcal{T}) \leq \Delta(|\lambda|(g(T_1), g(T_2), g(T_3), g(T_4))) = \mu^*(\lambda\mathcal{T}). \tag{3.6}$$

Thus, combining together Equations (3.5) and (3.6), we get

$$\mu^*(\lambda\mathcal{T}) = |\lambda|\mu^*(\mathcal{T}).$$

Hence, from (i) and (ii), it follows that μ^* is a semi-norm.

Furthermore, we have $\mu^*(\mathcal{T}) = 0$ if, and only if, $\Delta(g(T_1), g(T_2), g(T_3), g(T_4)) = 0$, by hypothesis if, and only if, $g(T_i) = 0$ for $i = 1, 2, 3, 4$.

Now, using the fact that g is a measure of non-strict-singularity, we have T_i is a strictly singular operator, for $i = 1, 2, 3, 4$. Moreover, when applying Proposition 1.1 (ii), we obtain $\mathcal{T}$ is a strictly singular operator.

Thus, we conclude that μ^* is a measure of non-strict-singularity on $X \times X$ of the operator $\mathcal{T}$. □

Now, as a result from Lemma 3.1, we have the following example.

Example 3.1 Let g be a measure of non-strict-singularity on a Banach space X, considering $\Delta(x_1, x_2, x_3, x_4) = \max\{(x_1 + x_2, x_3 + x_4)\}$ for any $(x_1, x_2, x_3, x_4) \in [0, +\infty[^4$, hence all the conditions of Lemma 3.1 are satisfied. Therefore,

$$\mu^*(\mathcal{T}) = \max\{g(T_1) + g(T_2), g(T_3) + g(T_4)\},$$

defines a measure of non-strict-singularity on a Banach space $X \times X$.

In what follows we make the following assumption :

$$\mathcal{H}: \begin{cases} g(HG(\mu)KG(\mu)) < \frac{1}{4} \quad g(HF(\mu)KF(\mu)) < \frac{1}{4}\\ g(HG(\mu)KF(\mu)) < \frac{1}{4} \quad g(HF(\mu)KG(\mu)) < \frac{1}{4}\\ \text{for some } \mu \in \rho(A) \text{ and all bounded operators } H \text{ and } K,\end{cases}$$

Remark 3.3 If the hypothesis $g(HF(\mu)KG(\mu)) < \frac{1}{4}$ holds for all bounded operators H and K, then $F(\mu)G(\mu)$ is strictly singular. Indeed, since $g(HF(\mu)KG(\mu)) < \frac{1}{4}$ is valid for all bounded operators H and K, we can consider $K = nI_X$, for all $n \in \mathbb{N}^*$ and $H = I_X$, we obtain $g(F(\mu)G(\mu)) < \frac{1}{4n}$. So, $g(F(\mu)G(\mu)) = 0$ and this implies that $F(\mu)G(\mu)$ is strictly singular. ◇

Theorem 3.5 Assume that L_0 defined in Equation (3.1) on a Banach space $X \times X$ satisfy $(H_1)-(H_5)$. *Let* $\mu \in \rho(A)$, and $\alpha \in [0,1]$. If $g(A) < 1$ and $g(S(\mu)) < 1$, then under hypothesis $(\mathcal{H})$, αL *is a generalized weakly demicompact operator for all* $\alpha \in [0,1]$. ⋄

Proof Let $\mu \in \rho(A)$ be such that hypothesis $(\mathcal{H})$ is satisfied and set λ be a complex number. It follows from Remark 3.2 that

$$\lambda I - L = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda I - A & 0 \\ 0 & \lambda I - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \mu) M(\mu),$$

where $M(\mu) = \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}$.

Let

$$\mathcal{K} = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix},$$

be a bounded operator on $X \times X$. Then,

$$\left[\mathcal{K} \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & 0 \end{pmatrix} \right]^2 = \begin{pmatrix} [K_2 F(\mu)]^2 + K_1 G(\mu) K_4 F(\mu) & K_2 F(\mu) K_1 G(\mu) + K_1 G(\mu) K_3 G(\mu) \\ K_4 F(\mu) K_2 F(\mu) + K_3 G(\mu) K_4 F(\mu) & K_4 F(\mu) K_1 G(\mu) + [K_3 G(\mu)]^2 \end{pmatrix}.$$

From hypothesis $(\mathcal{H})$ and by using measure μ^* defined in Example 3.1, it follows that

$$\mu^* \left((\lambda - \alpha\mu)^2 \left[\mathcal{K} \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & 0 \end{pmatrix} \right]^2 \right) < 1.$$

Which implies that, the operator

$$(\lambda - \alpha\mu) \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & 0 \end{pmatrix} \in \mathcal{I}_2(X).$$

Then, we can deduce from Remark 2.1 (ii) and the fact that $F(\mu)G(\mu)$ is strictly singular, that

$$(\lambda - \alpha\mu) \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix} \in \mathcal{I}_2(X).$$

Since $g(A) < 1$ and $g(S(\mu)) < 1$, and when applying Theorem 3.3, we deduce that αL is a generalized weakly demicompact operator for all $\alpha \in [0,1]$. □

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