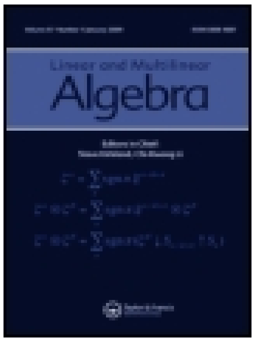

Linear and Multilinear Algebra

ISSN: 0308-1087 (Print) 1563-5139 (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

Spectral properties for generalized weakly *S*-demicompact operators

Imen Ferjani, Aref Jeribi & Bilel Krichen

LINEAR AND MULTILINEAR ALGEBRA
https://doi.org/10.1080/03081087.2020.1737633

Spectral properties for generalized weakly S-demicompact operators

Imen Ferjani, Aref Jeribi and Bilel Krichen

Faculty of Sciences of Sfax, Department of Mathematics, University of Sfax, Sfax, Tunisia

ABSTRACT
In this paper, we introduce the concept of generalized weakly S-demicompact operators with respect to a weakly closed linear operator S. We study the general setting of Fredholm theory. We give some perturbation results on the behaviour of relative essential spectra of the sum of two bounded linear operators by means of the relative essential spectra of each one. A characterization of generalized weakly S-demicompact operators by means of measures of weak noncompactness is given. Moreover, we impose some sufficient conditions on the inputs of a closable block operator matrix to ensure the generalized weak S-demicompactness of its closure.

ARTICLE HISTORY
Received 14 October 2019
Accepted 26 February 2020

COMMUNICATED BY
R. Pereira

KEYWORDS
Generalized weakly S-demicompact operator; Fredholm and semi-Fredholm operators; block operator matrix; relative essential spectrum

MATHEMATICS SUBJECT CLASSIFICATIONS
47A53; 47A10

1. Introduction

The concept of demicompactness appeared in the literature in order to construct and investigate the structure of fixed point sets for nonlinear operators acting on Hilbert and Banach spaces. It was introduced by Petryshyn in 1966 as follows:

Definition 1.1: An operator $T : \mathcal{D}(T) \subset X \longrightarrow X$ is said to be demicompact if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(x_n - Tx_n)_n$ converges to an element of X, there exists a convergent subsequence of $(x_n)_n$.

Later, this class is used by Petryshyn [1] and Akashi [2] to provide a few results on Fredholm theory. In 2015, Chaker et al. [3] continued this study to investigate the essential spectra of densely defined linear operators. In 2014, Krichen [4], introduced the relative demicompactness class with respect to a given closed linear operator as a generalization of the demicompactness notion. This definition asserts that if X is a Banach space, $T : \mathcal{D}(T) \subset X \longrightarrow X$, and $S : \mathcal{D}(S) \subset X \longrightarrow X$ are two linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S)$, then T is said to be S-demicompact, if every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(Sx_n - Tx_n)_n$ converges in X, have a convergent subsequence. In 2016, Krichen and

CONTACT Bilel Krichen krichen_bilel@yahoo.fr Faculty of Sciences of Sfax, Department of Mathematics, University of Sfax, Road Soukra Km 3.5, B.P. 1171, 3000 Sfax, Tunisia

O'Regan developed in [5] some Fredholm and perturbation results involving a new concept, called weakly relative demicompactness for nonlinear operators. Recently, in [6] the same authors studied the relationship between the class of weakly demicompact linear operators and an axiomatic measures of weak noncompactness of linear operator. In 2018, Jeribi et al. [7], characterized an unbounded S-demicompact linear operator with respect to a bounded linear operator S by the Kuratowski measure of noncompactness. However, in all these works, the study of spectral properties and perturbation results for block operator matrices with unbounded inputs is not treated.

Note that the theory of block operator matrices arise in various areas of mathematics and its applications. The spectral properties of block operator matrices are of vital importance as they govern for instance the time evolution and hence the stability of the underlying physical systems. From the most important works on the spectral theory of block operator matrices, we mention [8], in which the author developed the essential spectra of 2×2 and 3×3 block operator matrices. We also mention [9], which presented a wide panorama of methods to investigate the essential spectra of unbounded block operator matrices. In [10], it was proved that under some conditions, the block operator matrices is closable and we can determine its closure.

In the present paper, we will study some spectral properties for generalized weakly S-demicompact unbounded linear operators, where S is a bounded invertible linear operator. In particular, we discuss the case of block operator matrices, acting on a product Banach space $X \times X$, and having the following matrix representation:

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Note that the entries of L_0 are generally unbounded. The operator A acts on the Banach space X and has the domain $\mathcal{D}(A)$, D acts on the same Banach space X and is defined on $\mathcal{D}(D)$ and the intertwining operator B (resp. C) is defined on $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts from X into itself. In a part of our work, we aim to use the concept of generalized weak S-demicompactness to investigate the essential spectra of L, respecting a bounded invertible linear operator S.

Now let us recall some standard tools from Fredholm theory needed in this paper. Let X and Y be two Banach spaces. The set of all closed densely defined (resp. bounded) linear operators acting from X into Y is denoted by $\mathcal{C}(X, Y)$ (resp. $\mathcal{L}(X, Y)$). We denote by $\mathcal{K}(X, Y)$ (resp. $\mathcal{W}(X, Y)$ the subset of compact operators of (resp. the subset of weakly compact operators) $\mathcal{L}(X, Y)$. The set $\mathcal{W}(X, Y)$ is a closed two-sided ideal of $\mathcal{L}(X, Y)$ containing $\mathcal{K}(X, Y)$ (see [11, 12]).

Let X be a Banach space. We will use the following notation:

$\mathcal{C}(X)$ is the set of all closed densely defined linear operators in X acting to X. The sets $\mathcal{D}(T)$, $\mathcal{R}(T)$, and $\mathcal{N}(T)$ stand for the domain, the range and the kernel of $T \in \mathcal{C}(X)$ respectively.

$$\Phi_+(X) := \{T \in \mathcal{C}(X) : \ \mathcal{R}(T) \text{ is closed and } \alpha(T) = \dim \mathcal{N}(T) < \infty\},$$
$$\Phi_-(X) := \{T \in \mathcal{C}(X) : \ \mathcal{R}(T) \text{ is closed and } \beta(T) = \dim X/\mathcal{R}(T) < \infty\},$$
$$\Phi(X) := \Phi_-(X) \cap \Phi_+(X),$$

stand for the sets of upper semi-Fredholm, lower semi-Fredholm and Fredholm operators from X into itself.

For an operator $T \in \Phi_+(X)$ or $\Phi_-(X)$, the index is defined as $i(T) := \alpha(T) - \beta(T)$. If $T \in \mathcal{C}(X)$, we denote by $\rho(T)$ the resolvent set of T and by $\sigma(T)$ the spectrum of T.

Let $T \in \mathcal{C}(X)$. For $x \in \mathcal{D}(T)$, the graph norm $\|.\|_T$ of x is defined by $\|x\|_T = \|x\| + \|Tx\|$. It follows from the closedness of T that $X_T := (\mathcal{D}(T), \|.\|_T)$ is a Banach space. Clearly, for every $x \in \mathcal{D}(T)$ we have $\|Tx\| \leq \|x\|_T$, so that $T \in \mathcal{L}(X_T, X)$. A linear operator B is said to be T-defined if $\mathcal{D}(T) \subset \mathcal{D}(B)$. If the restriction of B to $\mathcal{D}(T)$ is bounded from X_T into X, we say that B is T-bounded. We can see T and B as bounded linear operators from X_T into X. In this case, they will be denoted by $\widehat{T}$ and $\widehat{B}$, respectively (that is, $\widehat{T} = T_{|X_T}$ and $\widehat{B} = B_{|X_T}$ are in $\mathcal{L}(X_T, X)$). Furthermore, we have the obvious relations:

(i) $\alpha(\widehat{T}) = \alpha(T), \beta(\widehat{T}) = \beta(T), \mathcal{R}(\widehat{B}) = \mathcal{R}(B)$,
(ii) $\alpha(\widehat{T} + \widehat{B}) = \alpha(T + B), \beta(\widehat{B} + \widehat{T}) = \beta(T + B)$, and
(iii) $\mathcal{R}(\widehat{B} + \widehat{T}) = \mathcal{R}(T + B)$.

Following these relations, we deduce that $T \in \Phi(X)$ (resp. $\Phi_+(X)$) if, and only if, $\widehat{T} \in \Phi(X_T, X)$ (resp. $\widehat{T} \in \Phi_+(X_T, X)$).

Definition 1.2: Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$. The operator F is called a:

(a) Fredholm perturbation if $T + F \in \Phi(X, Y)$ whenever $T \in \Phi(X, Y)$.
(b) Upper semi-Fredholm perturbation if $T + F \in \Phi_+(X, Y)$ whenever $T \in \Phi_+(X, Y)$.
(c) Lower semi-Fredholm perturbation if $T + F \in \Phi_-(X, Y)$ whenever $T \in \Phi_-(X, Y)$.

The set of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X, Y)$, $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$, respectively.

Now, we recall that the S-resolvent set of T is defined by:

$$\rho_S(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \text{ has a bounded inverse}\},$$

and the S-spectrum of T is defined by:

$$\sigma_S(T) := \mathbb{C} \backslash \rho_S(T).$$

Next, we denote by asc(T) (resp. desc(T)) the ascent (resp. the descent) of T, i.e. the smallest nonnegative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$(resp. $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$). If such n does not exists, then asc$(T) = \infty$ (resp. desc$(T) = \infty$). For $\lambda \in \mathbb{C}$, an eigenvalue of T, the dimension of $\mathcal{N}^\infty(\lambda - T) := \cup_{n \in \mathbb{N}} \mathcal{N}((\lambda - T)^n)$ is called the algebraic multiplicity of T and is denoted by mult(T, λ). An operator $T \in \mathcal{L}(X)$ is called Browder if it is Fredholm and has finite ascent and finite descent. An operator $T \in \mathcal{L}(X)$ is called upper semi-Browder if it is upper semi-Fredholm and has finite ascent. The class of all Browder operators is defined by:

$$\mathcal{B}(X) := \{T \in \Phi(X) \text{ such that asc } (T) < \infty \text{ and desc } (T) < \infty\}.$$

The class of all upper semi-Browder operators on a Banach space X that is defined by:

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) \text{ such that asc } (T) < \infty\}.$$

The classes of operators above defined motivate the definition of several spectra (see for example [13–15]). Note that the concept of S-essential spectrum was introduced in [16] as a generalization of the usual notion of Wolf essential spectrum (see [17]).

In this work, we are concerned with the following S-essential spectra: The upper semi-Browder S-spectrum of $T \in \mathcal{L}(X)$ is defined by:

$$\sigma_{ub,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \mathcal{B}_+(X)\},$$

whilst the Browder S-spectrum of $T \in \mathcal{L}(X)$ is defined by:

$$\begin{aligned}
&\sigma_{b,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \mathcal{B}(X)\}.\\
&\sigma_{e_1,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \Phi_+(X)\} := \mathbb{C}\backslash\Phi_{+T,S},\\
&\sigma_{e_4,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \Phi(X)\} := \mathbb{C}\backslash\Phi_{T,S},\\
&\sigma_{e_5,S}(T) := \mathbb{C}\backslash\rho_{5,S}(T),\\
&\sigma_{e_6,S}(T) := \mathbb{C}\backslash\rho_{6,S}(T),\\
&\sigma_{ap,S}(T) := \left\{\lambda \in \mathbb{C} \text{ such that } \inf_{x\in D(T),\ \|x\|=1} \|(\lambda S - T)x\| = 0\right\},
\end{aligned}$$

where $\rho_{5,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda \in \Phi_{T,S} \text{ and } i(\lambda S - T) = 0\}$ and $\rho_{6,S}(T)$ denote the set of those $\lambda \in \rho_{5,S}(T)$ such that all scalars near λ are in $\rho_S(T)$.

We organize the paper in the following way: In the next section, we recall some definitions and results needed in the sequel of the paper. In Section 3, we discuss the relationship between generalized weakly S-demicompact operators and Fredholm and upper semi-Fredholm operators. Moreover, a characterization by means of measure of weak noncompactness is given. In Section 4, we provide some sufficient conditions on the inputs of a closable block operator matrix to ensure the generalized weak S-demicompactness of its closure. Also, we establish some perturbation results for S-essential spectrum of operator matrices.

2. Preliminary results

The notion of the measure of weak noncompactness has many applications in topology, functional analysis and theories of differential and integral equations (see [18–20]). It was introduced by De Blasi. In order to recall this notion, let us denote by X a Banach space, by B_X the open unit ball of X and by $\overline{B}_X$ its closure. The family of all nonempty and bounded subsets of X will be denoted by $\mathcal{M}_X$ and $\mathcal{W}_X$ its subfamily consisting of all relatively weakly compact sets. Moreover, let $conv(A)$ denotes the convex hull of a set $A \subset X$. Let us recall the following definition:

Definition 2.1 ([21]): Let X be a Banach space. A function $\mu : \mathcal{M}_X \longrightarrow [0, +\infty[$ is said to be a measure of weak noncompactness in X if, for all $A, B \in \mathcal{M}_X$, it satisfies the following conditions:

(i) $\mu(A) = 0$ if, and only if, A is relatively weakly compact ($A \in \mathcal{W}_X$).
(ii) If $A \subset B$, then $\mu(A) \leq \mu(B)$.
(iii) $\mu(\overline{\text{conv}(A)}) = \mu(A)$.
(iv) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.
(v) $\mu(A + B) \leq \mu(A) + \mu(B)$.
(vi) $\mu(\lambda A) = |\lambda|\mu(A)$, for $\lambda \in \mathbb{C}$.

As an example of regular measure in a Banach space X, we have the measure of weak noncompactness defined by De Blasi (see [22]) in the following formula: for all $A \in \mathcal{M}_X$,

$$w(A) = \inf\left\{t > 0, \text{ there exists } C \in \mathcal{W}_X \text{ such that } A \subset C + t\overline{B}_X\right\}.$$

Definition 2.2: Let X and Y be two complex Banach spaces and w be the De Blasi measure of weak noncompactness in the space Y. We define the function

$$\begin{aligned} \Theta_w : \mathcal{L}(X, Y) &\longrightarrow [0, +\infty[\\ T &\longmapsto \Theta_w(T) = w(T(\overline{B}_X)). \end{aligned}$$

Θ_w is called a measure of weak noncompactness of operators associated with w.

Proposition 2.1: *Let X and Y be two complex Banach spaces and $T \in \mathcal{L}(X, Y)$. Then,*

(i) $w(T(F)) \leq \Theta_w(T)w(F)$ *for every* $F \in \mathcal{M}_X$.
(ii) $\Theta_w(T) \leq \|T\|$.
(iii) *If* $X = Y$, *then* $\Theta_w(T^n) \leq (\Theta_w(T))^n$ *for every* $n \in \mathbb{N}$.
(iv) *Let* $C \geq 0$ *such that for every* $x \in X$, $\|Tx\| \leq C\|x\|$. *Then,*

$$w\left(T(F)\right) \leq Cw(F).$$

(v) Let $C \geq 0$ *such that for every* $x \in X$, $\|x\| \leq C\|Tx\|$. *Then,*

$$w(F) \leq Cw\left(T(F)\right).$$

Lemma 2.1 ([23]): *Let X be a Banach space and $A \in \mathcal{L}(X)$. An operator A is in $\Phi_+(X)$ with $i(A) \leq 0$ if, and only if, there exist two operators A_0 and K such that A_0 is in $\Phi_+(X)$ and one to one, and K is a finite rank operator such that $A = A_0 + K$.*

Definition 2.3: Let X and Y be two Banach spaces.

(i) An operator $T \in \mathcal{C}(X, Y)$ is said to have a left Fredholm inverse if there exists $T_l \in \mathcal{L}(Y, X_T)$ such that $I_{X_T} - T_l\widehat{T} \in \mathcal{K}(X_T)$.
(ii) An operator $T \in \mathcal{C}(X, Y)$ is said to have a right Fredholm inverse if there exists $T_r \in \mathcal{L}(Y, X_T)$ such that $I_Y - \widehat{T}T_r \in \mathcal{K}(Y)$.

Theorem 2.1: *Let X, Y, and Z be Banach spaces, $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$. Then*

(i) *If* $AB \in \Phi_+(X, Z)$, *then* $B \in \Phi_+(X, Y)$.
(ii) *If* $AB \in \Phi_-(X, Z)$, *then* $A \in \Phi_-(Y, Z)$.

Theorem 2.2: *Let X, Y, and Z be Banach spaces, $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$. If A and B are Fredholm operators, upper semi-Fredholm operators, lower semi-Fredholm operators, then AB is Fredholm operator, upper semi-Fredholm operator, lower semi-Fredholm operator, respectively, and*

$$i(AB) = i(A) + i(B).$$

Definition 2.4 ([24]): Let X be a Banach space and $A \in \mathcal{L}(X)$. A is called a generalized Riesz operator if there exists a finite subset E of $\mathbb{C}$ such that

(i) For all $\lambda \in \mathbb{C}\backslash E$, $A - \lambda I \in \Phi(X)$,
(ii) for all $\lambda \in \mathbb{C}\backslash E$, $A - \lambda I$ has a finite ascent and finite descent, and
(iii) all $\lambda \in \sigma(A)\backslash E$, are eigenvalues of finite multiplicity and have no accumulation point except possibly points of E.

Corollary 2.1 ([25]): *$\lambda \in \sigma_{ap,S}(T)\backslash\sigma_{ub,S}(T)$ if, and only if, λ is an isolated point of $\sigma_{ap,S}(T)$, an eigenvalue of T of finite multiplicity,* $\text{asc}(T - \lambda S) < \infty$ *and $\mathcal{R}(T - \lambda S)$ is closed.*

Corollary 2.2 ([26]): *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then,*

$$\sigma_{ub,S}(T) = \sigma_{e_1,S}(T) \cup acc\ \sigma_{ap,S}(T),$$

where $acc\ \sigma_{ap,S}(T)$ is the set of accumulation points of $\sigma_{ap,S}(T)$.

Definition 2.5: Let X be a Banach space, $T : \mathcal{D}(T) \subset X \longrightarrow X$ and let $S : \mathcal{D}(S) \subset X \longrightarrow X$ be densely defined linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S)$. T is said to be weakly S-demicompact (or weak relative demicompact with respect to S), if every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(Sx_n - Tx_n)_n$ converges weakly in X, it has a weakly convergent subsequence.

3. Generalized weak *S*-demicompactness

In the rest of this paper, we denote $\to$ for the strong convergence and $\rightharpoonup$ for the weak convergence.

Now, we introduce the following definition:

Definition 3.1: Let X be a Banach space and let $A, S \in \mathcal{C}(X)$ with $\mathcal{D}(A) \subset \mathcal{D}(S)$. A is called a generalized weakly S-demicompact operator if there exists a finite subset E of $\mathbb{C}$ containing 0 such that:

(i) For all $\lambda \in \mathbb{C}\backslash E$, $1/\lambda A$ is weakly S-demicompact operator,
(ii) for all $\lambda \in \mathbb{C}\backslash E$, $\lambda S - A$ has a finite ascent, and
(iii) all $\lambda \in \sigma_S(A)\backslash E$, are eigenvalues of finite multiplicity and have no accumulation points except possibly points of E.

The set E is called a generalized set of A.

Remark 3.1: (i) It should be noted that if, E is a generalized set of A and G is a finite subset of $\mathbb{C}$ containing E, then G is also a generalized set of A.

(ii) Note that for $S = I$, we recover the usual definition of generalized weak demicompactness of an operator introduced by I. Ferjani et al. in [27].

Now, we will show that there is a relationship between the class of generalized weakly S-demicompact operators and upper semi-Fredholm operators acting on a Banach space.

Theorem 3.1: *Let X be a Banach space, $T \in \mathcal{C}(X)$ and $S \in \mathcal{L}(X)$ such that $0 \in \rho(S)$, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$ and $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$. Then, T is a generalized weakly S-demicompact if, and only if, there exists a finite subset $E \subset \mathbb{C}$ containing 0 such that $\lambda S - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$.*

Proof: Suppose that, there exists a finite subset $E \subset \mathbb{C}$ containing 0 such that $\lambda S - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$. Since S is an invertible operator, we can write

$$\lambda S - T = S(\lambda I - S^{-1}T) \tag{1}$$

Taking into account Equation (1) and apply Theorem 2.1, we infer that $\lambda I - S^{-1}T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$. Thus, $S^{-1}T$ is a generalized Riesz operator. Hence, according to Definition 2.4, we have all $\lambda \in \sigma(S^{-1}T)\backslash E$, are eigenvalues of finite multiplicity and have no accumulation points except possibly points of E.

Now, the fact that S is bounded and invertible allows us to get the following equality:

$$\sigma(S^{-1}T) = \sigma_S(T).$$

Consequently, we deduce that all $\lambda \in \sigma_S(T)\backslash E$, are eigenvalues of finite multiplicity and have no accumulation points except possibly points of E.

Now, let us check that $\mathrm{asc}(\lambda S - T)$ is finite. Since $S^{-1}T$ is a generalized Riesz operator, it follows that $\mathrm{asc}(\lambda I - S^{-1}T) < \infty$ for all $\lambda \in \mathbb{C}\backslash E$.

We denote by $n_0 = \mathrm{asc}(\lambda I - S^{-1}T) < \infty$. Further, for any eigenvalues $\lambda \in \sigma(S^{-1}T)\backslash E$, we have $\mathrm{mult}(S^{-1}T, \lambda) < \infty$ which implies that

$$\mathrm{mult}(S^{-1}T, \lambda) = \dim \mathcal{N}^{\infty}(\lambda I - S^{-1}T) = \dim \mathcal{N}((\lambda I - S^{-1}T)^{n_0}) < \infty.$$

We can write

$$\lambda I - S^{-1}T = S^{-1}(\lambda S - T). \tag{2}$$

Let $n \in \mathbb{N}$, using Equation (2), we have

$$(\lambda I - S^{-1}T)^n = (S^{-1})^n(\lambda S - T)^n$$

Hence,

$$\mathcal{N}((\lambda S - T)^n) \subset \mathcal{N}((\lambda I - S^{-1}T)^n), \text{ for all } n \in \mathbb{N}.$$

This implies that

$$\bigcup_{n\in\mathbb{N}} \mathcal{N}((\lambda S - T)^n) \subset \bigcup_{n\in\mathbb{N}} \mathcal{N}((\lambda I - S^{-1}T)^n)$$

Now, since $\dim \bigcup_{n\in\mathbb{N}} \mathcal{N}((\lambda I - S^{-1}T)^n) = \dim \mathcal{N}((\lambda I - S^{-1}T)^{n_0}) < \infty$, it follows that $\dim \bigcup_{n\in\mathbb{N}} \mathcal{N}((\lambda S - T)^n) < \infty$, hence we get $\mathrm{asc}(\lambda S - T) < \infty$, for all $\lambda \in \mathbb{C}\backslash E$.

Now for the last point, using the fact that $\operatorname{asc}(\lambda S - T) < \infty$, we obtain that $i(\lambda S - T) \leq 0$, then, in view of Lemma 2.1, there exists a bounded below operator T_0 and $K \in \mathcal{K}(X)$ such that

$$\lambda S - T = T_0 + K.$$

Let $(x_n)_n$ be a bounded sequence in $\mathcal{D}(T)$ such that

$$(\lambda S - T)x_n \rightharpoonup x \in X.$$

Then, $((T_0 + K)x_n)_n$ converges weakly on X. Since K compact, it follows that $(Kx_n)_n$ has a convergent subsequence $(Kx_{\varphi(n)})_n$.

Consequently, $(T_0x_{\varphi(n)})_n$ is a weakly convergent sequence. Since T_0 is bounded below, we infer that there exists a positive constant C such that

$$\|x\| \leq C\|T_0x\|,$$

for all $x \in \mathcal{D}(T)$.

Let $F = \{x_n,\ n \in \mathbb{N}\}$, then by using Proposition 2.1 we get

$$w(F) \leq Cw\left(T_0(F)\right).$$

Thus, $w(F) = 0$ and therefore, $(x_{\varphi(n)})_n$ has a weakly convergent subsequence. Hence, we get $(1/\lambda)T$ is weakly S-demicompact for all $\lambda \in \mathbb{C}\backslash E$. Thus, T is a generalized weakly S-demicompact operator.

To prove the converse, let us first check that $\alpha(\lambda S - T) < \infty$. Since T is a generalized weakly S-demicompact operator, it follows that $\operatorname{asc}(\lambda S - T) < \infty$ for all $\lambda \in \mathbb{C}\backslash E$. We denote by $p = \operatorname{asc}(\lambda S - T) < \infty$. Furthermore, for any eigenvalue $\lambda \in \sigma_S(T)\backslash E$, we have $\operatorname{mult}(T, \lambda S) < \infty$ which implies that

$$\operatorname{mult}(T, \lambda S) = \dim \mathcal{N}^{\infty}(\lambda S - T) = \dim \mathcal{N}((\lambda S - T)^p) < \infty.$$

Hence, from the following inclusion

$$\mathcal{N}(\lambda S - T) \subset \mathcal{N}((\lambda S - T)^p),$$

we get $\dim \mathcal{N}(\lambda S - T) = \alpha(\lambda S - T) < \infty$.

For the second part of this proof, we claim that $\mathcal{R}(\lambda S - T)$ is closed. Let $(x_n)_n$ be any sequence included in $\mathcal{D}(T)$. Let us take $y_n = (\lambda S - T)x_n$ such that $y_n \to y \in X$. Since $(1/\lambda)T$ is a weakly S-demicompact and

$$Sx_n - \frac{1}{\lambda}Tx_n \rightharpoonup \frac{1}{\lambda}y \in X,$$

there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that $x_{\varphi(n)} \rightharpoonup x \in X$.

Using both

$$x_{\varphi(n)} \rightharpoonup x \quad \text{and} \quad (\lambda S - T)x_{\varphi(n)} \rightharpoonup y,$$

together with the closedness of $(\lambda S - T)$, we infer that

$$x \in \mathcal{D}(T) \quad \text{and} \quad y = (\lambda S - T)x.$$

As result, $y \in \mathcal{R}(\lambda S - T)$. Hence, $\mathcal{R}(\lambda S - T)$ is closed. ∎

Remark 3.2: It seems that the hypotheses S is invertible cannot be dropped. Indeed, let consider the backward shift $A : l^2 \to l^2$, $A(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ and define the following two operators acting from $l^2 \times l^2$ to itself:

$$S = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Then, $\lambda S - T \in \Phi_+(l^2 \times l^2)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. According to the above theorem T should be generalized weakly S-demicompact, and in particular it should admit a nonnegative integer n such that $\mathcal{N}((\lambda S - T)^n) = \mathcal{N}((\lambda S - T)^{n+1})$ (the ascent of $\lambda S - T$ is finite). However, it easy to see that $\dim \mathcal{N}((\lambda S - T)^k) = k$ for all $k \geq 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$, so the kernel equality can never happen.

The following result will show the connection between Fredholm operators having null indices, and the class of generalized weakly S-demicompact operators.

Theorem 3.2: *Let X be a Banach space, $T \in \mathcal{C}(X)$ and $S \in \mathcal{L}(X)$ such that $0 \in \rho(S)$, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$ and $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$. If μT is a generalized weakly S-demicompact operator for each $\mu \in [0, 1]$ with a generalized subset E, then $\lambda S - T \in \Phi(X)$ and $i(\lambda S - T) = i(\lambda S)$, for all $\lambda \in \mathbb{C}\backslash E$.*

Proof: Since μT is a generalized weakly S-demicompact operator for each $\mu \in [0, 1]$, then $\lambda S - \mu T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$. Moreover, from Remark 2.4 in [4], it follows that the index $i(\lambda S - \mu T)$ is continuous in μ and since it is integer, including infinite values, it must be constant for every $\mu \in [0, 1]$. Showing that $i(\lambda S - \mu T) = i(\lambda S - T) = i(\lambda S)$. Consequently, $\lambda S - T \in \Phi(X)$ and $i(\lambda S - T) = i(\lambda S)$, for all $\lambda \in \mathbb{C}\backslash E$. ■

Let us consider the following assumption for a given operator T.

($\mathcal{H}$) If $(x_n)_n$ is weakly convergent sequence in X, then $(Tx_n)_n$ has a strongly convergent subsequence in X.

Lemma 3.1: *Let X be a Banach space, $T \in \mathcal{C}(X)$ and $S \in \mathcal{L}(X)$ such that $0 \in \rho(S)$. If T is weakly S-demicompact satisfying the assumption ($\mathcal{H}$), then $S - T$ is an upper semi-Fredholm operator.*

Proof: First, let us check that $\mathcal{N}(S - T)$ is finite dimensional. Let

$$\Omega := \{x \in \mathcal{D}(T) \text{ such that } (S - T)x = 0 \text{ and } \|x\| \leq 1\},$$

and let $(x_n)_n$ be a sequence of Ω. As T is weakly S-demicompact satisfying assumption ($\mathcal{H}$), there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that $x_{\varphi(n)} \rightharpoonup x \in X$ and $Tx_{\varphi(n)} \to y$. Thus, it follows from the boundedness of S^{-1} that $x_{\varphi(n)} \to S^{-1}y = x$. Now, by using the closedness of T, we deduce that $x \in \mathcal{D}(T)$ and $Tx = y$. Consequently, $x \in \mathcal{N}(S - T)$. Moreover, from the fact that $x_{\varphi(n)} \rightharpoonup x$, we get $\|x\| \leq \liminf \|x_{\varphi(n)}\| \leq 1$. Hence, $x \in \Omega$ and so $\alpha(S - T) < \infty$.

Now, we have to prove that $\mathcal{R}(S - T)$ is closed. Let $(x_n)_n$ be any sequence included in $\mathcal{D}(T)$. Let us take $y_n = (S - T)x_n$ such that $y_n \to y \in X$. Since T is a weakly S-demicompact and $Sx_n - Tx_n \rightharpoonup y \in X$, there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$

such that $x_{\varphi(n)} \rightharpoonup x \in X$. Using both $x_{\varphi(n)} \rightharpoonup x$ and $(S-T)x_{\varphi(n)} \rightharpoonup y$, together with the closedness of $(S-T)$, we infer that

$$x \in \mathcal{D}(T) \quad \text{and} \quad y = (S-T)x.$$

As result, $y \in \mathcal{R}(S-T)$. Hence, $\mathcal{R}(S-T)$ is closed. This achieves the proof. ■

Now, we give a characterization of generalized weakly S-demicompact operators by means of a measure of weak noncompactness.

Theorem 3.3: *Let X be a Banach space, $T \in \mathcal{C}(X)$ and $S \in \mathcal{L}(X)$ such that $0 \in \rho(S)$, $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$ and $S(\mathcal{D}(T)) \subset \mathcal{D}(T)$. Assume that T satisfies the assumption $(\mathcal{H})$. Then, T is a generalized weakly S-demicompact if, and only if, there exist a positive constant C and a finite subset E of $\mathbb{C}$ containing 0 such that for all bounded sets $F \subset \mathcal{D}(T)$*

$$w(F) \leq Cw\left((\lambda S - T)(F)\right), \quad \text{for all } \lambda \in \mathbb{C} \backslash E.$$

Proof: Suppose that T is a generalized weakly S-demicompact, then by using Theorem 3.1, there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda S - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$. Now, since $\operatorname{asc}(\lambda S - T)$ is finite, it follows that $i(\lambda S - T) \leq 0$, applying Lemma 2.1, there exist a compact operator K and a bounded below operator T_0 such that $\lambda S - T = K + T_0$. Since T_0 is bounded below, there exists a positive constant C such that

$$\|x\| \leq C\|T_0 x\|,$$

for all $x \in \mathcal{D}(T)$.

Hence, by applying Proposition 3.1, we get

$$w(F) \leq Cw\left((\lambda S - T)(F)\right),$$

for any bounded set $F \subset \mathcal{D}(T)$.

To prove the converse, suppose that there exist a positive constant C and a finite subset E of $\mathbb{C}$ containing 0 such that for every bounded set F of X,

$$w(F) \leq Cw\left((\lambda S - T)(F)\right), \text{ for all } \lambda \in \mathbb{C}\backslash E.$$

Let $(x_n)_n$ be a bounded sequence of $\mathcal{D}(T)$ such that

$$(\lambda S - T)x_n \rightharpoonup x \in X.$$

Choose $F = \{x_n;\ n \in \mathbb{N}\}$. It is clear that F is a bounded set of $\mathcal{D}(T)$ such that $w((\lambda S - T)(F)) = 0$. Hence, $w(F) = 0$ which follows that $(x_n)_n$ has a weakly convergent subsequence. Consequently, $(1/\lambda)T$ is weakly S-demicompact for all $\lambda \in \mathbb{C}\backslash E$. Now, since T satisfies $(\mathcal{H})$, it follows from Lemma 3.1 that $\lambda S - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$. Thus, applying Theorem 3.1, T is a generalized weakly S-demicompact operator. ■

Corollary 3.1: *Let X be a Banach space, $T \in \mathcal{L}(X)$, $A \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X)$ such that $0 \in \rho(S)$. Suppose that there exists an integer $n \geq 1$, such that $(AT)^n$ or $(TA)^n$ satisfy the assumption $(\mathcal{H})$ and $AT - TA \in \mathcal{W}(X)$. Then, $(AT)^n$ is a generalized weakly S-demicompact if, and only if, $(TA)^n$ is a generalized weakly S-demicompact.*

Proof: Suppose that $(AT)^n$ is a generalized weakly S-demicompact. In the view of Theorem 3.3, there exist a positive constant C_n and a finite subset E of $\mathbb{C}$ containing 0 such as for all $F \in \mathcal{M}_X$,

$$w(F) \leq C_n w\left((\lambda S - (AT)^n)(F)\right), \text{ for all } \lambda \in \mathbb{C}\backslash E.$$

Hence, for every bounded set F,

$$w(F) \leq C_n w\left((\lambda S - (TA)^n)(F)\right) + C_n w\left(((TA)^n - (AT)^n)(F)\right).$$

Using the following identity:

$$(TA)^n - (AT)^n = \sum_{k=0}^{n-1} (TA)^k (TA - AT)(AT)^{n-1-k},$$

we infer that $(TA)^n - (AT)^n \in \mathcal{W}(X)$. Hence, for all $F \in \mathcal{M}_X$,

$$w\left(((TA)^n - (AT)^n)(F)\right) = 0.$$

Accordingly, for all $F \in \mathcal{M}_X$,

$$w(F) \leq C_n w\left((\lambda S - (TA)^n)(F)\right), \text{ for all } \lambda \in \mathbb{C}\backslash E.$$

In view of Theorem 3.3, we conclude that $(TA)^n$ is a generalized weakly S-demicompact. The converse is proved similarly. ■

Lemma 3.2: *Let X be a Banach space, $T \in \mathcal{C}(X)$ and $S \in \mathcal{L}(X)$ such that $S \neq 0$. Assume that $S - \widehat{T}$ satisfies the assumption $(\mathcal{H})$. If T is a S-demicompact operator, then T is a weakly S-demicompact operator.*

Proof: Let $(x_n)_n$ be a bounded sequence of $\mathcal{D}(T)$ such that $y_n := Sx_n - Tx_n \rightharpoonup y$. Since T is a S-demicompact operator, by using Theorem 2.1 in [7], we deduce that $S - T \in \Phi_+(X)$ which implies that $S - \widehat{T} \in \Phi_+(X_T, X)$. The use of $\alpha(S - T) < \infty$, implies that there exists a closed subspace X_0 of X such that

$$X = \mathcal{N}(S - T) \oplus X_0.$$

Since X_0 is a closed subset of X and $T \in \mathcal{C}(X)$, we deduce that $X_0 \cap \mathcal{D}(T)$ is a closed subset of X_T. Therefore $(X_0 \cap \mathcal{D}(T), \|.\|_T)$ is a Banach space. Hence,

$$X_T = \mathcal{N}(S - \widehat{T}) \oplus [\mathcal{D}(T) \cap X_0].$$

Then, $(S - \widehat{T})_{|X_0 \cap \mathcal{D}(T)} : (X_0 \cap \mathcal{D}(T), \|.\|_T) \longrightarrow (\mathcal{R}(S - T), \|.\|)$ is invertible with bounded inverse on $\mathcal{R}(S - T) = \mathcal{R}(S - \widehat{T})$. Then from the boundedness of $((S - \widehat{T})_{|X_0 \cap \mathcal{D}(T)})^{-1}$ on $\mathcal{R}(S - \widehat{T})$, we have $x_n \rightharpoonup (S - \widehat{T})^{-1}(y)$. Taking into account the fact that $S - \widehat{T}$ satisfies $(\mathcal{H})$, we deduce that $(S - \widehat{T})x_{\varphi(n)} \to x$, so $(S - T)x_{\varphi(n)} \to x$. From the relative demicompactness of T, there exists a convergent subsequence $(x_{\psi(\varphi(n))})_n$ of $(x_{\varphi(n)})_n$ such that $x_{\psi(\varphi(n))} \to z$. Hence, $x_{\psi(\varphi(n))} \rightharpoonup z$ and this achieves the proof. ■

Theorem 3.4: *Let X be a Banach space, $A \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X)$ such that $0 \in \rho(S)$ and $0 \in \sigma_{ub,S}(A)$. Assume that $\lambda S - A$ satisfies the assumption $(\mathcal{H})$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. Then, A is a generalized weakly S-demicompact if, and only if, A has a finite upper semi-Browder S-spectrum.*

Proof: Suppose that A is a generalized weakly S-demicompact. Then, from Theorem 3.1, there exists a finite set $E \subset \mathbb{C}$ containing 0 such that for all $\lambda \in \mathbb{C}\backslash E$, $A - \lambda S \in \Phi_+(X)$ and have a finite ascent. Therefore $\sigma_{ub,S}(A) \subset E$ and hence $\sigma_{ub,S}(A)$ is a finite subset.

To prove the converse, suppose that $\sigma_{ub,S}(A)$ is a finite subset of $\mathbb{C}$. For all $\lambda \in \mathbb{C}\backslash\sigma_{ub,S}(A)$, $A - \lambda S$ is an upper semi-Browder, then $A - \lambda S$ is an upper semi-Fredholm which implies from Theorem 2.1 in [7] that $(1/\lambda)A$ is a S-demicompact for all $\lambda \in \mathbb{C}\backslash\sigma_{ub,S}(A)$, so from Lemma 3.2, we infer that $(1/\lambda)A$ is a weakly S-demicompact for all $\lambda \in \mathbb{C}\backslash\sigma_{ub,S}(A)$. Furthermore, we have $\operatorname{asc}(A - \lambda S) < \infty$, for all $\lambda \in \mathbb{C}\backslash\sigma_{ub,S}(A)$. Let $\lambda \in \sigma_S(A)\backslash\sigma_{ub,S}(A)$. Since $\lambda \in \sigma_S(A)$, by using Corollary 2.2, λ is an isolated point of $\sigma_{ap,S}(A)$. Hence, $\lambda \in \sigma_{ap,S}(A)\backslash\sigma_{ub,S}(A)$. Now, applying Corollary 2.1, we deduce that λ is an eigenvalue of A of finite multiplicity. Again, by using Corollary 2.2, we get all $\lambda \in \sigma_S(A)\backslash\sigma_{ub,S}(A)$ have no accumulation point except possibly points of $\sigma_{ub,S}(A)$. Therefore, A is a generalized weakly S-demicompact with a generalized set $\sigma_{ub,S}(A)$. ■

In the end of this section, we will give some perturbation results and the relationship between the S-essential spectrum of the sum of two linear operators and the S-essential spectrum of each one.

Theorem 3.5: *Let X be a Banach space, $T \in \mathcal{L}(X)$, $A \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X)$ such that $S \neq 0$ and $0 \in \rho(S)$. Assume that there exists a finite subset E of $\mathbb{C}$ containing 0 such that the following assertions hold :*

(i) *For every $\lambda \in \Phi_{+(T+A),S}\backslash E$, there exists $F_{\lambda l}$ a left Fredholm inverse operator of $(\lambda S - T - A)$ such that $-\Upsilon_\lambda(T,A)F_{\lambda l}$ is a generalized weakly S-demicompact with a generalized set E.*
(ii) *For every $\lambda \in \Phi_{+(T+A),S}\backslash E$, there exists $H_{\lambda l}$ a left Fredholm inverse operator of $(\lambda S - T - A)$ such that $-\Upsilon_\lambda(A,T)H_{\lambda l}$ is a generalized weakly S-demicompact with a generalized set E.*

Then,

$$[\sigma_{e_1,S}(T) \cup \sigma_{e_1,S}(A)]\backslash E \subset [\sigma_{e_1,S}(T+A)]\backslash E,$$

where $\Upsilon_\lambda(T,A) = TA - \lambda[T,S]$ and $[T,S] = TS - ST$.

Proof: Let $\lambda \in \mathbb{C}\backslash E$. If there exists $F_{\lambda l}$ a left Fredholm inverse operators of $(\lambda S - T - A)$, then $F_{\lambda l}(\lambda S - T - A) = I - K$ where $K \in \mathcal{K}(X)$, then we have

$$\begin{aligned}(\lambda S - T)(\lambda S - A) &= \lambda S(\lambda S - T - A) + \Upsilon_\lambda(T,A)\\ &= \lambda S(\lambda S - T - A) + \Upsilon_\lambda(T,A)F_{\lambda l}(\lambda S - T - A) + \Upsilon_\lambda(T,A)K.\end{aligned}$$

We conclude

$$(\lambda S - T)(\lambda S - A) = (\lambda S + \Upsilon_\lambda(T, A)F_{\lambda l})(\lambda S - T - A) + \Upsilon_\lambda(T, A)K. \tag{3}$$

In the same manner, we can write

$$(\lambda S - A)(\lambda S - T) = (\lambda S + \Upsilon_\lambda(A, T)H_{\lambda l})(\lambda S - T - A) + \Upsilon_\lambda(A, T)K. \tag{4}$$

Let $\lambda \notin \sigma_{e_1,S}(T + A) \cup E$, then $\lambda S - T - A \in \Phi_+(X)$ and set $F_{\lambda l}$ and $H_{\lambda l}$ are left Fredholm inverse operators of $(\lambda S - T - A)$ such that $-\Upsilon_\lambda(T, A)F_{\lambda l}$ and $-\Upsilon_\lambda(A, T)H_{\lambda l}$ are generalized weakly S-demicompact. Now, applying Theorem 3.1, we get $(\lambda S + \Upsilon_\lambda(T, A)F_{\lambda l})$ and $(\lambda S + \Upsilon_\lambda(A, T)H_{\lambda l})$ are upper semi-Fredholm operators on X, for all $\lambda \in \mathbb{C}\backslash E$. Using Theorem 2.2, Equation (3) and (4), we prove that $(\lambda S - T)(\lambda S - A)$ and $(\lambda S - A)(\lambda S - T)$ are upper semi-Fredholm operators both. Now, Theorem 2.1 completes the proof. ■

Theorem 3.6: *Let X be a Banach space, $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(X)$, $S \in \mathcal{L}(X)$ such that $S \neq 0$, $0 \in \rho(S)$ and let E be a finite subset of $\mathbb{C}$ containing 0.*

(i) *If for every $\lambda \in \Phi_{+(A+B),S}\backslash E$, there exists $F_{\lambda l}$ (resp. $F_{\lambda r}$) a left (resp. right) Fredholm inverse operator of $(\lambda S - A - B)$ such that $\mu(\lambda[A, S] - AB)F_{\lambda l}$ (resp. $\mu F_{\lambda r}(\lambda[S, B] - AB)$) is a generalized weakly S-demicompact with a generalized set E for any $\mu \in [0, 1]$, then*

$$\sigma_{e_4,S}(A + B)\backslash E \subset [\sigma_{e_4,S}(A) \cup \sigma_{e_4,S}(B)]\backslash E.$$

(ii) *Moreover, if there exists $H_{\lambda l}$ (resp. $H_{\lambda r}$) a left (resp. right) Fredholm inverse operator of $(\lambda S - A - B)$ such that $\mu(\lambda[B, S] - BA)H_{\lambda l}$ (resp. $\mu H_{\lambda r}(\lambda[S, A] - BA)$) is a generalized weakly S-demicompact with a generalized set E for any $\mu \in [0, 1]$, then*

$$\sigma_{e_4,S}(A + B)\backslash E = [\sigma_{e_4,S}(A) \cup \sigma_{e_4,S}(B)]\backslash E,$$

where $[A, S] = AS - SA$.

Proof: (*i*) Let $\lambda \in \mathbb{C}\backslash E$. If there exists $F_{\lambda l}$ (*resp.*$F_{\lambda r}$) a left (resp. right) Fredholm inverse operator of $(\lambda S - A - B)$, then there exists $K \in \mathcal{K}(X)$ (*resp.*$K' \in \mathcal{K}(X)$) such that $F_{\lambda l}(\lambda S - A - B) = I - K$ (resp. $(\lambda S - A - B)F_{\lambda r} = I - K'$). Thus, we have

$$\begin{aligned}(\lambda S - A)(\lambda S - B) &= \lambda S(\lambda S - A - B) + \Theta_\lambda(A, B)\\ &= \lambda S(\lambda S - A - B) + \Theta_\lambda(A, B)F_{\lambda l}(\lambda S - A - B) + \Theta_\lambda(A, B)K\\ &= (\lambda S + \Theta_\lambda(A, B)F_{\lambda l})(\lambda S - A - B) + \Theta_\lambda(A, B)K,\end{aligned}$$

where $\Theta_\lambda(A, B) = AB - \lambda[A, S]$.

(resp. $(\lambda S - A)(\lambda S - B) = (\lambda S - A - B)(\lambda S + F_{\lambda r}(\lambda[B, S] + AB)) + K'(\lambda[B, S] + AB)$.)

Let $\lambda \notin [\sigma_{e_4,S}(A) \cup \sigma_{e_4,S}(B)] \cup E$, then $\lambda S - A \in \Phi(X)$ and $\lambda S - B \in \Phi(X)$. It follows from Theorem 6 in [26] that $(\lambda S - A)(\lambda S - B) \in \Phi(X)$. As $\mu([A, S] - AB)F_{\lambda l}$ (resp. $\mu F_{\lambda r}(\lambda[S, B] - AB)$) is a generalized weakly S-demicompact, it follows from Theorem 3.2 that $(\lambda S + \Theta_\lambda(A, B)F_{\lambda l})$ (resp. $(\lambda S + F_{\lambda r}(\lambda[B, S] + AB))$) is a Fredholm operator. Now, using the fact that $\Theta_\lambda(A,B)K$ (resp. $K'(\lambda[B,S]+AB)$) is compact, we conclude together with Theorem 7.12 in [28] that $\lambda S - A - B \in \Phi(X)$. Hence, $\lambda \notin \sigma_{e_4,S}(A+B)$.

(ii) Conversely, if $\lambda \notin \sigma_{e_4,S}(A+B) \cup E$, then arguing the operators as above, we deduce that $(\lambda S - A)(\lambda S - B) \in \Phi(X)$ and $(\lambda S - B)(\lambda S - A) \in \Phi(X)$. According to Theorem 5.14 in [28], we conclude that $\lambda S - A \in \Phi(X)$ and $\lambda S - B \in \Phi(X)$.

Thus, $\lambda \notin [\sigma_{e_4,S}(A) \cup \sigma_{e_4,S}(B)] \backslash E$. ■

4. Generalized weak S-demicompactness results for block operator matrices

The purpose of this section is to discuss the S-essential spectra of the closure L of the 2×2 operator matrix L_0 which acts on the Banach space $X \times X$, where S is a bounded invertible operator, formally defined on the product space $X \times X$ by a matrix:

$$S := \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix},$$

and L_0 is given by

$$L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{5}$$

The operator A acts on X and has domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space X, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp., $\mathcal{D}(C)$) and acts on X.

In what follows, we will assume that the following conditions hold:

($H1$) The operator A is a closed, densely defined linear operator on X with a nonempty S_1-resolvent set $\rho_{S_1}(A)$.
($H2$) The operator B is a densely defined linear operator on X and, for some (hence for all) $\mu \in \rho_{S_1}(A)$, the operator $(A-\mu S_1)^{-1}B$ is closable. (In particular, if B is closable, then $(A-\mu S_1)^{-1}B$ is also closable).
($H3$) The operator C satisfies the inclusion $\mathcal{D}(A) \subset \mathcal{D}(C)$ and, for some (hence for all) $\mu \in \rho_{S_1}(A)$, the operator $C(A-\mu S_1)^{-1}$ is bounded.
($H4$) The lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in X and for some (hence all) $\mu \in \rho_{S_1}(A)$, the operator $D - C(A-\mu S_1)^{-1}B$ is closable. We will denote the closure of the operator $D - (C-\mu S_3)(A-\mu S_1)^{-1}(B-\mu S_2)$ by $S(\mu)$.

Remark 4.1: (i) From the closed graph theorem, it follows that the operator $G(\mu) := (A-\mu S_1)^{-1}(B-\mu S_2)$ is bounded on X.

(ii) From the assumption ($H3$), it follows that the operator: $F(\lambda) = (C-\mu S_3)(A-\mu S_1)^{-1}$ is bounded on X.

Now, let us recall the following result which describes the closure of the operator L_0.

Theorem 4.1 ([8]): *Let conditions ($H1$)–($H3$) be satisfied and the lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ be dense in X. Then, the operator L_0 is closable if, and only if, the operator $D - C(A-\mu S_1)^{-1}B$*

is closable on X, for some $\mu \in \rho_{S_1}(A)$. Moreover, the closure L of L_0 is given by:

$$L = \mu S + \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} A - \mu S_1 & 0 \\ 0 & S(\mu) - \mu S_4 \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}.$$

Remark 4.2: Let $\lambda \in \mathbb{C}$ and $\mu \in \rho_{S_1}(A)$. Writing $\lambda S - L = \mu S - L + (\lambda - \mu)S$ and using Theorem 4.1, we have

$$\begin{aligned} \lambda S - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda S_1 - A & 0 \\ 0 & \lambda S_4 - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \mu) M(\mu) \\ &:= UV(\lambda)W - (\lambda - \mu)M(\mu), \end{aligned} \tag{6}$$

where

$$M(\mu) = \begin{pmatrix} 0 & S_1 G(\mu) - S_2 \\ F(\mu) S_1 - S_3 & F(\mu) S_1 G(\mu) \end{pmatrix}.$$

In all that follows, we will consider the following invertible matrix operator:

$$S := \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \in \mathcal{L}(X \times X), \tag{7}$$

such that $0 \in \rho(S_1) \cap \rho(S_4)$.

Now, we start this section by a result for a particular representation of L_0.

Lemma 4.1: *Let X be a Banach space, $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(X)$ and consider the 2×2 operator matrices*

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X \times X)$ with the representation (7)*. Assume that $S_3 \in \mathcal{F}(X)$. Then, A is a generalized weakly S_1-demicompact operator and B is a generalized weakly S_4-demicompact operator if, and only if, M_C is a generalized weakly S-demicompact operator.*

Proof: Let $\lambda \in \mathbb{C}$. Clearly, we have

$$\lambda S - M_C = \begin{pmatrix} \lambda S_1 - A & \lambda S_2 - C \\ \lambda S_3 & \lambda S_4 - B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \lambda S_3 & 0 \end{pmatrix} + \begin{pmatrix} \lambda S_1 - A & \lambda S_2 - C \\ 0 & \lambda S_4 - B \end{pmatrix}.$$

Since A is a generalized weakly S_1-demicompact and B is a generalized weakly S_4-demicompact, then there exist two finite subsets E_1 and E_2 of $\mathbb{C}$ containing 0 such that $\lambda S_1 - A \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E_1$ and $\lambda S_4 - B \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E_2$. From Remark 3.1, it follows that also A is a generalized weakly S_1-demicompact and B is a generalized weakly S_4-demicompact with a generalized set $E = E_1 \cup E_2$. This allows us to get $\lambda S_1 - A \in \Phi_+(X)$ and $\lambda S_4 - B \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$. Now, when applying Lemma 6.6.1 in [8] and using the fact that $S_3 \in \mathcal{F}(X)$, we get $\lambda S - M_C \in \Phi_+(X \times X)$ for all $\lambda \in \mathbb{C}\backslash E$, and for every $C \in \mathcal{L}(X)$. Hence, from Theorem 3.1, M_C is a generalized weakly S-demicompact with a generalized set E.

To prove the converse, assume that M_C is a generalized weakly S-demicompact operator. From Theorem 3.1, there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda S - M_C$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C}\backslash(E \cup \sigma_{S_4}(B))$.

First, we put the following factorization:

$$\begin{aligned}\lambda S - M_C &= \begin{pmatrix} 0 & 0 \\ \lambda S_3 & 0 \end{pmatrix} + \begin{pmatrix} \lambda S_1 - A & \lambda S_2 - C \\ 0 & \lambda S_4 - B \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \lambda S_3 & 0 \end{pmatrix} + \begin{pmatrix} I & (\lambda S_2 - C)(\lambda S_4 - B)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda S_1 - A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \lambda S_4 - B \end{pmatrix}.\end{aligned}$$

Since we have found that

$$\begin{pmatrix} \lambda S_1 - A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \lambda S_4 - B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \lambda S_4 - B \end{pmatrix} \begin{pmatrix} \lambda S_1 - A & 0 \\ 0 & I \end{pmatrix},$$

so using the facts that $\lambda S - M_C \in \Phi_+(X \times X)$ and $S_3 \in \mathcal{F}(X)$, it follows from Lemma 6.6.2 in [8] that $\lambda S_1 - A$ and $\lambda S_4 - B$ are upper semi-Fredholm, for all $\lambda \in \mathbb{C}\backslash E$. Consequently, by using Theorem 3.1, the operators A is a generalized weakly S_1-demicompact and B is a generalized weakly S_4-demicompact. ■

Now, rewrite the Frobenius–Schur factorization:

$$\alpha L = \alpha\mu S - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \alpha\mu S_1 - \alpha A & 0 \\ 0 & \alpha\mu S_4 - \alpha S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}.$$

Let $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\lambda S - \alpha L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda S_1 - \alpha A & 0 \\ 0 & \lambda S_4 - \alpha S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \alpha\mu)M(\mu) \\ &:= UV_\alpha(\lambda)W - (\lambda - \alpha\mu)M(\mu).\end{aligned}$$

Theorem 4.2: *Let X be a Banach space and $S \in \mathcal{L}(X \times X)$ with the representation* (7). *Assume that the operator L_0 defined in Equation* (5) *satisfies* (H_1)–(H_4) *and let L be its closure. Let $\mu \in \rho_{S_1}(A)$ and $\lambda \in \mathbb{C}$. If for $t \in [0, 1]$, the operators tA is a generalized weakly S_1-demicompact, $tS(\mu)$ is a generalized weakly S_4-demicompact and $M(\mu) \in \mathcal{F}_+(X \times X)$, then αL is a generalized weakly S-demicompact operator, for all $\alpha \in [0, 1]$.*

Proof: Let $\mu \in \rho_{S_1}(A)$, $\alpha \in [0, 1]$ and $\lambda \in \mathbb{C}$. According to Equation (6), we have

$$\lambda S - \alpha L = UV_\alpha(\lambda)W - (\lambda - \alpha\mu)M(\mu).$$

Since αA is a generalized weakly S_1-demicompact and $\alpha S(\mu)$ is a generalized weakly S_4-demicompact, then there exist two finite subsets E_1 and E_2 of $\mathbb{C}$ containing 0 such that $\lambda S_1 - \alpha A \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E_1$ and $\lambda S_4 - \alpha S(\mu) \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E_2$. From Remark 3.1, it follows that also αA is a generalized weakly S_1-demicompact and $\alpha S(\mu)$ is a generalized weakly S_4-demicompact with a generalized set $E = E_1 \cup E_2$. This allows us to get $\lambda S_1 - \alpha A \in \Phi_+(X)$ and $\lambda S_4 - \alpha S(\mu) \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$. Now, when applying Lemma 6.6.1 in [8], we get $V_\alpha(\lambda) \in \Phi_+(X \times X)$ for all $\lambda \in \mathbb{C}\backslash E$. Taking into account the fact that $M(\mu) \in \mathcal{F}_+(X \times X)$ and the boundedness of the operators U, W and theirs inverses, we deduce that $\lambda S - \alpha L \in \Phi_+(X \times X)$ for all $\lambda \in \mathbb{C}\backslash E$. Hence, from Theorem 3.1, αL is a generalized weakly S-demicompact operator with a generalized set E. ■

Corollary 4.1: *Let X be a Banach space and $S \in \mathcal{L}(X \times X)$ with the representation* (7). *Assume that the operator L_0 defined in Equation* (5) *satisfies* (H_1)-(H_4) *and let L be its closure. Let $\mu \in \rho_{S_1}(A)$ and $\lambda \in \mathbb{C}$. If A is a generalized weakly S_1-demicompact operator and $S(\mu)$ is a generalized weakly S_4-demicompact operator and $M(\mu) \in \mathcal{F}_+(X \times X)$, then L is a generalized weakly S-demicompact operator.*

Proof: The proof is a direct application of Theorem 4.2 for $\alpha = 1$. ■

Theorem 4.3: *Let X be a Banach space and $S \in \mathcal{L}(X \times X)$ with the representation* (7). *Assume that there exists a finite subset E of $\mathbb{C}$ containing 0, the operator L_0 defined in Equation* (5) *satisfies* (H_1)-(H_4) *and let L be its closure. Then*

(i) *If the operators A is a generalized weakly S_1-demicompact, $S(\mu)$ is a generalized weakly S_4-demicompact with a generalized set E and $M(\mu) \in \mathcal{F}_+(X \times X)$, then*

$$\sigma_{e_{1,S}}(L)\backslash E = [\sigma_{e_{1,S_1}}(A) \cup \sigma_{e_{1,S_4}}(S(\mu))]\backslash E.$$

(ii) *If for all $\alpha \in [0, 1]$, the operators αA is a generalized weakly S_1-demicompact, $\alpha S(\mu)$ is a generalized weakly S_4-demicompact with a generalized set E and $M(\mu) \in \mathcal{F}(X \times X)$, then*

$$\sigma_{e_{i,S}}(L)\backslash E = [\sigma_{e_{i,S_1}}(A) \cup \sigma_{e_{i,S_4}}(S(\mu))]\backslash E, \text{ where } i \in \{4, 5\}.$$

Proof: (i) Since A is a generalized weakly S_1-demicompact and $S(\mu)$ is a generalized weakly S_4-demicompact and $M(\mu) \in \mathcal{F}_+(X \times X)$, it follows from Corollary 4.1 that the matrix operator L is a generalized weakly S-demicompact. Hence, from Theorem 3.1, there exists a finite subset E of $\mathbb{C}$ containing 0 such that $\lambda S - L$ is an upper semi-Fredholm operator for all $\lambda \in \mathbb{C}\backslash E$.

According to Remark 4.2, we have

$$\begin{aligned}\lambda S - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda S_1 - A & 0 \\ 0 & \lambda S_4 - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \mu) M(\mu) \\ &:= UV(\lambda)W - (\lambda - \mu)M(\mu),\end{aligned}$$

where

$$M(\mu) = \begin{pmatrix} 0 & S_1 G(\mu) - S_2 \\ F(\mu)S_1 - S_3 & F(\mu)S_1G(\mu) \end{pmatrix}.$$

Using the fact that $M(\mu) \in \mathcal{F}_+(X \times X)$, we infer that $\lambda S - L \in \Phi_+(X \times X)$ if, and only if, the operator $UV(\lambda)W$ is such too. Now, since V and W are invertible and have bounded inverses, hence $\lambda S - L \in \Phi_+(X \times X)$ if, and only if, $V(\lambda) \in \Phi_+(X \times X)$ which is equivalent to $\lambda S_1 - A \in \Phi_+(X)$ and $\lambda S_4 - S(\mu) \in \Phi_+(X)$. Thus,

$$\sigma_{e_{1,S}}(L)\backslash E = [\sigma_{e_{1,S_1}}(A) \cup \sigma_{e_{1,S_4}}(S(\mu))]\backslash E.$$

(ii) Since αA is a generalized weakly S_1-demicompact, $\alpha S(\mu)$ is a generalized weakly S_4-demicompact operators for all $\alpha \in [0, 1]$ and $M(\mu) \in \mathcal{F}(X \times X)$, it follows from Theorem 4.2 that the matrix operator αL is a generalized weakly S-demicompact for all $\alpha \in [0, 1]$. Hence, from Theorem 3.2, $\lambda S - L \in \Phi(X \times X)$.

Now, a similar reasoning as (i) allows us to conclude that

$$\sigma_{e_{4,S}}(L)\backslash E = [\sigma_{e_{4,S_1}}(A) \cup \sigma_{e_{4,S_4}}(S(\mu))]\backslash E.$$

For Schechter's spectrum. Since αA is a generalized weakly S_1-demicompact and $\alpha S(\mu)$ is a generalized weakly S_4-demicompact for all $\alpha \in [0, 1]$, it follows from Theorem 4.2 that the matrix operator αL is a generalized weakly S-demicompact for all $\alpha \in [0, 1]$. Hence, from Theorem 3.2, $\lambda S - L \in \Phi(X \times X)$ and $i(\lambda S - L) = 0$.

Now, when using the same argument of (i), we infer that

$$\sigma_{e_{5,S}}(L)\backslash E = [\sigma_{e_{5,S_1}}(A) \cup \sigma_{e_{5,S_4}}(S(\mu))]\backslash E. \qquad \blacksquare$$

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

[1] Petryshyn WV. Remarks on condensing and k-set-contractive mappings. J Math Anal Appl. 1972;39(3):717–741.
[2] Akashi WY. On the perturbation theory for Fredholm operators. Osaka J Math. 1984;21: 603–612.
[3] Chaker W, Jeribi A, Krichen B. Demicompact linear operators, essential spectrum and some perturbation results. Math Nachr. 2015;288(13):1476–1486.
[4] Krichen B. Relative essential spectra involving relative demicompact unbounded linear operators. Acta Math Sci Ser B Engl Ed. 2014;34:546–556.
[5] Krichen B, O'Regan D. On the class of relatively weakly demicompact nonlinear operators. Fixed Point Theory. 2018;19:625–630.
[6] Krichen B, O'Regan D. Weakly demicompact linear operators and axiomatic measures of weak noncompactness. Math Slovaca. 2019;69:1403–1412.
[7] Jeribi A, Krichen B, Salhi M. Characterization of relatively demicompact operators by means of measures of noncompactness. J Korean Math Soc. 2018;55(3):877–895.
[8] Jeribi A. Spectral theory and applications of linear operators and block operator matrices. New York: Springer-Verlag; 2015.
[9] Tretter C. Spectral theory of block operator matrices and applications. London: Copyright by Imperial College Press; 2008.
[10] Atkinson FV, Langer H, Mennicken R, et al. The essential spectrum of some matrix operators. Math Nachr. 1994;167:5–20.
[11] Dunford N, Schwartz JT. Linear operators, Part I. General theory. New York: Interscience; 1958.
[12] Goldberg S. Unbounded linear operators, theory and applications. New York: McGraw-Hill Book Co; 1966.
[13] Gustafson K, Weidmann J. On the essential spectrum. J Math Anal Appl. 1969;25:121–127.
[14] Jeribi A. Linear operators and their essential pseudospectra. Boca Raton: CRC Press; 2018.

[15] Schechter M. On the essential spectrum of an arbitrary operator. J Math Anal Appl. 1966;13:205–215.
[16] Faierman M, Mennicken R, Moller M. A boundary eigenvalue problem for a system of partial differential operators occuring in magnetohydrodynamics. Math Nachr. 1995;173:141–167.
[17] Wolf F. On the invariance of the essential spectrum under a change of boundary conditions of partial differential boundary operators. Nederl Akad Wetensch Proc Ser A 62 Indag Math. 1959;21:142–147.
[18] Astala K. On measures of noncompactness and ideal variations in Banach spaces. Ann Acad Sci Fenn Ser AI Math Diss. 1980;29:1–42.
[19] Emmanuele G. Measure of weak noncompactness and fixed point theorems. Boll Math Soc Sci Math RS Roumanie. 1981;25:353–358.
[20] Jeribi A, Krichen B. Nonlinear functional analysis in Banach spaces and Banach algebras: fixed point theory under weak topology for nonlinear operators and block operator matrices with applications. CRC Press Taylor and Francis; 2015. (Monographs and Research Notes in Mathematics).
[21] Banaś J, Martinón A. On measures of weak noncompactness in Banach sequence spaces. Portugal Math. 1995;52:131–138.
[22] De Blasi FS. On a property of the unit sphere in a Banach space. Bull Math Soc Sci Math R S Roumanie (NS). 1977;21:259–262.
[23] Williams V. Closed Fredholm and Semi-Fredholm Operators, essential spectra and perturbations. J Funct Anal. 1975;20:1–25.
[24] Jeribi A, Moalla N. Fredholm operators and Riesz theory for polynomially compact operators. Acta Appl Math. 2006;90:227–245.
[25] Rakočević V. Approximate point spectrum and commuting compact perturbations. Glasgow Math J. 1986;28:193–198.
[26] Müller V. Spectral theory of linear operators and spectral systems in Banach algebras. Second. Germany: Birkhäuser- Verlag; 2007.
[27] Ferjani I, Jeribi A, Krichen B. Spectral properties involving generalized weakly demicompact operators. Mediterr J Math Art. 2019;21:30.
[28] Schechter M. Principles of functional analysis. New York: Academic Press; 1971.