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Spectral Properties Involving Generalized Weakly Demicompact Operators

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Abstract. In this paper, we investigate the concept of generalized weakly demicompact operators with respect to weakly closed densely defined linear operators. We give their relationship with Fredholm and upper semi-Fredholm operators. In particular a characterization by means of upper semi-Browder spectrum is given. Moreover, we provide some sufficient conditions on the inputs of a closable block operator matrix to ensure the generalized weak demicompactness of its closure. Our results generalize many known ones in the literature.

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1. Introduction

Let X and Y be two Banach spaces. The set of all closed densely defined (resp. bounded) linear operators acting from X into Y is denoted by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$). We denote by $\mathcal{K}(X,Y)$ the subset of compact operators of $\mathcal{L}(X,Y)$. For $T \in \mathcal{C}(X,Y)$, we use the following notations: $\alpha(T)$ is the dimension of the kernel $\mathcal{N}(T)$ and $\beta(T)$ is the codimension of the range $\mathcal{R}(T)$ in Y. The next sets of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from X into Y are, respectively, defined by:

$$\Phi_+(X,Y) := \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$

$$\Phi_-(X,Y) := \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$

$$\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y),$$

and

$$\Phi_\pm(X,Y) := \Phi_-(X,Y) \cup \Phi_+(X,Y).$$

For $T \in \Phi_\pm(X,Y)$, the index is defined as $i(T) := \alpha(T) - \beta(T)$. A complex number λ is in $\Phi_{+T}, \Phi_{-T}, \Phi_{\pm T}$ or Φ_T if $\lambda - T$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_\pm(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$,

$\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$, respectively. If $T \in \mathcal{C}(X)$, we denote by $\rho(T)$ the resolvent set of T and by $\sigma(T)$ the spectrum of T. Let $T \in \mathcal{C}(X)$. For $x \in \mathcal{D}(T)$, the graph norm $\|.\|_T$ of x is defined by $\|x\|_T = \|x\| + \|Tx\|$. It follows from the closedness of T that $X_T := (\mathcal{D}(T), \|.\|_T)$ is a Banach space. Clearly, for every $x \in \mathcal{D}(T)$ we have $\|Tx\| \leq \|x\|_T$, so that $T \in \mathcal{L}(X_T, X)$. A linear operator B is said to be T-defined if $\mathcal{D}(T) \subseteq \mathcal{D}(B)$. If the restriction of B to $\mathcal{D}(T)$ is bounded from X_T into X, we say that B is T-bounded.

Definition 1.1. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$. The operator F is called a:

(a) Fredholm perturbation if $T + F \in \Phi(X, Y)$ whenever $T \in \Phi(X, Y)$.
(b) Upper semi-Fredholm perturbation if $T + F \in \Phi_+(X, Y)$ whenever $T \in \Phi_+(X, Y)$.
(c) Lower semi-Fredholm perturbation if $T + F \in \Phi_-(X, Y)$ whenever $T \in \Phi_-(X, Y)$. ♢

The set of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations is denoted by $\mathcal{F}(X, Y), \mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$, respectively.
Let $\operatorname{asc}(T)$(resp. $\operatorname{desc}(T)$) denote the ascent (resp. the descent) of T, i.e., the smallest non-negative integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$(resp. $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$). If such n does not exist, then $\operatorname{asc}(T) = \infty$ (resp. $\operatorname{desc}(T) = \infty$). An operator $T \in \mathcal{L}(X)$ is called a Weyl operator if it is Fredholm of index zero. An operator $T \in \mathcal{L}(X)$ is Browder if it is Fredholm and has finite ascent and finite descent. It is well known that every Browder operator is Weyl. An operator $T \in \mathcal{L}(X)$ is called upper semi-Browder if it is upper semi-Fredholm and has finite ascent. The class of all Browder operators is defined by:

$$\mathcal{B}(X) := \{T \in \Phi(X) \text{ such that } \operatorname{asc}(T) < \infty \text{ and } \operatorname{desc}(T) < \infty\}.$$

The class of all upper semi-Browder operators on a Banach space X is defined by:

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) \text{ such that } \operatorname{asc}(T) < \infty\}.$$

The classes of operators defined above motivate the definition of several spectra. The upper semi-Browder spectrum of $T \in \mathcal{L}(X)$ is defined by:

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \mathcal{B}_+(X)\},$$

whilst the Browder spectrum of $T \in \mathcal{L}(X)$ is defined by:

$$\sigma_b(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \mathcal{B}(X)\}.$$

For $\lambda \in \mathbb{C}$, an eigenvalue of T, the $\dim \mathcal{N}^\infty(\lambda - T)$ is called the algebraic multiplicity of T and denoted by $\operatorname{mult}(T, \lambda)$. An operator $T \in \mathcal{L}(X)$ is said to be weakly compact, if $T(M)$ is relatively weakly compact for every bounded subset $M \subseteq X$. The family of weakly compact operators on X, is denoted by $\mathcal{W}(X)$. Moreover, $\mathcal{W}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (see [13, 15]). A Banach space X is said to have the Dunford–Pettis property ([12]) (in short term DPP) if, for each Banach space Y, every weakly compact operator $T : X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets in Y. An operator $T \in \mathcal{L}(X)$ is said to be a Dunford–Pettis operator, if T maps weakly compact sets into compact sets.

The concept of demicompactness appeared in the literature to discuss fixed points. It was introduced by Petryshyn in 1966 as follows:

Definition 1.2. An operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be demicompact if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $x_n - Tx_n \to x \in X$, there exists a convergent subsequence of $(x_n)_n$. ♢

We will denote by $\mathcal{DC}(X)$, the family of demicompact operators on X. Clearly, $\mathcal{K}(X) \subset \mathcal{DC}(X)$.

Note that this class is used by Petryshyn [25] and Akashi [1] to provide a few results on Fredholm theory. Recently, Chaker et al. [8] continued this study to investigate the essential spectra of densely defined linear operators. In 2014, Krichen [20], introduced the relative demicompactness class with respect to a given closed linear operator as a generalization of the demicompactness notion. In 2016, Krichen and O'Regan developed in [21] some Fredholm and perturbation results involving the class of weakly demicompact linear operators. Moreover, they studied the relationship between this class and measures of weak noncompactness of linear operator with respect to an axiomatic one

The theory of block operator matrices arises in various areas of mathematics and its applications: in systems theory as Hamiltonians (see [10]), in the discretization of partial differential equations as large partitioned matrices due to sparsity patterns, in saddle point problems in non-linear analysis (see [6]), in evolution problems as linearization of second-order Cauchy problems and as linear operators describing coupled systems of partial differential equations. Such systems occur widely in mathematical physics, e.g., in fluid mechanics (see [9]), magnetohydrodynamics (see [23]) and quantum mechanics (see [29]). In all these applications, the spectral properties of the corresponding block operator matrices are of vital importance as they govern, for instance, the time evolution and hence the stability of the underlying physical systems. From the most important works on the spectral theory of block operator matrices, we mention [16], in which the author developed the essential spectra of 2×2 and 3×3 block operator matrices. We also mention [30], in which was presented a wide panorama of methods to investigate the essential spectra of block operator matrices. In this paper, we will study the generalized weak demicompactness properties of the following matrix operator L_0 acting on the Banach space product $X \times X$ which is defined by:

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

In general, the entries of L_0 are unbounded. The operator A acts on the Banach space X and has the domain $\mathcal{D}(A)$, D acts on the same Banach space X and is defined on $\mathcal{D}(D)$ and the intertwining operator B (resp. C) is defined on $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts from X into itself. Below, we shall assume that $\mathcal{D}(A) \subset \mathcal{D}(C)$ and $\mathcal{D}(B) \subset \mathcal{D}(D)$. Note in general that the operator L_0 is neither closed nor closable operator even if its entries are closed operators. In [3] it was proved that under some conditions, L_0 is closable and its closure is denoted by L. In the literature, many important results were obtained concerning the spectral theory of this type of operators. We mention from these works the book [30] in which the author investigated the essential spectra of the matrix operator by means of an abstract non-zero two-sided ideal. The aim of this work is to use the concept of generalized weak demicompactness to investigate the essential spectra of L, the closure of L_0.

In this work, we are concerned with the following essential spectra:

$$\begin{aligned}
&\sigma_{e_1}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_+(X)\} := \mathbb{C}\backslash\Phi_{+T},\\
&\sigma_{e_4}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi(X)\} := \mathbb{C}\backslash\Phi_{T},\\
&\sigma_{e_5}(T) := \mathbb{C}\backslash\rho_5(T),\\
&\sigma_{e_6}(T) := \mathbb{C}\backslash\rho_6(T),\\
&\sigma_{ap}(T) := \left\{\lambda \in \mathbb{C} \text{ such that} \inf_{x\in D(T);\|x\|=1} \|(\lambda - T)x\| = 0\right\},
\end{aligned}$$

where $\rho_5(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda \in \Phi_T \text{ and } i(\lambda - T) = 0\}$ and $\rho_6(T)$ denotes the set of those $\lambda \in \rho_5(T)$ such that all scalars near λ are in $\rho(T)$. The subsets $\sigma_{ap}(\cdot)$, $\sigma_{e_1}(\cdot)$ and $\sigma_{e_4}(\cdot)$ are, respectively, the approximate point spectrum, the Gustafson and the Wolf essential spectra. $\sigma_{e_5}(\cdot)$ is the Schechter essential spectrum.

We organize the paper in the following way: in the next section, we recall some definitions and results needed in the sequel of the paper. In Sect. 3, we discuss the relationship between generalized weakly demicompact operators and Fredholm and upper semi-Fredholm operators. Moreover, a characterization by mean of measure of noncompactness is given. In Sect. 4, we provide some sufficient conditions on the inputs of a closable block operator matrix to ensure the generalized weak demicompactness of its closure. Also, we establish some perturbation results for essential spectrum of matrix operators.

2. Preliminary Results

The notion of the measure of weak noncompactness has many applications in topology, functional analysis and theories of differential and integral equations (see [2,14,18]). It was introduced by De Blasi. To recall this notion, let us denote by X a Banach space, by B_X the open unit ball of X and by $\overline{B}_X$ its closure. The family of all nonempty and bounded subsets of X will be denoted by $\mathcal{M}_X$ and $\mathcal{W}_X$ its subfamily consisting of all relatively weakly compact sets. Moreover, let $\operatorname{conv}(A)$ denote the convex hull of a set $A \subset X$. Note that there exists an axiomatic approach in defining measures of weak noncompactness. Let us recall the following definition:

Definition 2.1 [4]. Let X be a Banach space. A function $\mu : \mathcal{M}_X \longrightarrow [0, +\infty[$ is said to be a measure of weak noncompactness in X if, for all $A, B \in \mathcal{M}_X$, it satisfies the following conditions:

(i) $\mu(A)=0$ if, and only if, A is relatively weakly compact ($A \in \mathcal{W}_X$).
(ii) If $A \subset B$, then $\mu(A) \leq \mu(B)$.
(iii) $\mu(\overline{\mathrm{conv}(A)}) = \mu(A)$.
(iv) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.
(v) $\mu(A+B) \leq \mu(A) + \mu(B)$.
(vi) $\mu(\lambda A) = |\lambda|\mu(A)$, for $\lambda \in \mathbb{C}$. $\diamondsuit$

Let us recall that each measure of weak noncompactness also satisfies the Cantor intersection condition (see [5]), if $X_n \in \mathcal{W}_X$, $X_n := \overline{X_n}$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and if $\lim_{n\to+\infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^{+\infty} X_n \neq \emptyset$.

As an example of regular measure in a Banach space X, we have the measure of weak noncompactness defined by De Blasi (see [11]) in the following formula: for all $A \in \mathcal{M}_X$,

$$w(A) = \inf\left\{t > 0, \text{there exists a set } C \in \mathcal{W}_X \text{ such that } A \subset C + t\overline{B}_X\right\}.$$

Now, we provide a definition which gives an axiomatic approach to the notion of measure of weak noncompactness of operators.

Definition 2.2. Let X, Y be two Banach spaces, and let μ be a measure of weak noncompactness in Y. We define the function

$$\begin{aligned} \Psi_\mu : \mathcal{L}(X,Y) &\longrightarrow [0, +\infty[\\ T \quad &\longmapsto \Psi_\mu(T) = \mu(T(\overline{B}_X)). \end{aligned}$$

Ψ_μ is called a measure of weak noncompactness of operators associated to μ. $\diamondsuit$

In view of this definition, we get the following properties of the function Ψ_μ.

Proposition 2.1. *Let X, Y be two Banach spaces, let μ be a measure of weak noncompactness in Y and let Ψ_μ be a measure of weak noncompactness of operators associated to μ. For all $A, T \in \mathcal{L}(X,Y)$, we have:*

(i) $\Psi_\mu(T) = 0$ *if, and only if, T is weakly compact.*
(ii) *If S is a bounded subset of X, then $\mu(T(S)) \leq \Psi_\mu(T)\mu(S)$.*
(iii) $\Psi_\mu(A+T) \leq \Psi_\mu(A) + \Psi_\mu(T)$.
(iv) $\Psi_\mu(\lambda A) = |\lambda|\Psi_\mu(A)$ *for all $\lambda \in \mathbb{C}$.*
(v) $\Psi_\mu(AT) \leq \Psi_\mu(A).\Psi_\mu(T)$.
(vi) $\Psi_\mu(A+K) = \Psi_\mu(A)$ *for all $K \in \mathcal{W}(X,Y)$.* $\diamondsuit$

The following definitions can be found in [17].

Definition 2.3. Let X be a Banach space. An operator $A \in \mathcal{L}(X)$ is called a Riesz operator if

(i) For all $\lambda \in \mathbb{C}\backslash\{0\}$, $A - \lambda \in \Phi(X)$,
(ii) for all $\lambda \in \mathbb{C}\backslash\{0\}$, $A - \lambda$ has a finite ascent and finite descent, and
(iii) all $\lambda \in \sigma(A)\backslash\{0\}$, are eigenvalues of finite multiplicity and have no accumulation point except possibly zero. $\diamondsuit$

The set of Riesz operators acting on X is denoted by $\mathcal{R}(X)$.

Definition 2.4. Let X be a Banach space. An operator $A \in \mathcal{L}(X)$ is called a generalized Riesz operator if there exists E a finite subset of $\mathbb{C}$ such that

(i) For all $\lambda \in \mathbb{C}\backslash E$, $A - \lambda \in \Phi(X)$,
(ii) for all $\lambda \in \mathbb{C}\backslash E$, $A - \lambda$ has a finite ascent and finite descent, and
(iii) all $\lambda \in \sigma(A)\backslash E$, are eigenvalues of finite multiplicity and have no accumulation point except possibly points of E. $\diamondsuit$

Lemma 2.1 [32]. *Let X be a Banach space, $A \in \mathcal{L}(X)$ and $S \in \mathcal{R}(X)$. Then*
(i) *If $A \in \Phi(X)$ and $AS - SA \in \mathcal{F}(X)$, then $A + S \in \Phi(X)$ and $i(A+S) = i(A)$.*
(ii) *If $A \in \Phi_+(X)$ and $AS - SA \in \mathcal{F}_+(X)$, then $A + S \in \Phi_+(X)$ and $i(A+S) = i(A)$.*
(iii) *If $A \in \Phi_-(X)$ and $AS - SA \in \mathcal{F}_-(X)$, then $A + S \in \Phi_-(X)$ and $i(A+S) = i(A)$.* $\diamondsuit$

We say that an operator $A \in \mathcal{L}(X)$ is a polynomially Riesz if, there exists a non-zero complex polynomial P such that $P(A) \in \mathcal{R}(X)$ and also there exists a unique non-zero complex polynomial m_A with leading coefficient 1 and of the minimal degree such that $m_A \in \mathcal{R}(X)$ which we call the minimal polynomial of A. These operators have been recently discussed in [33].

Corollary 2.1 [26]. *$\lambda \in \sigma_{ap}(T)\backslash\sigma_{ab}(T)$ if, and only if, λ is an isolated point of $\sigma_{ap}(T)$, an eigenvalue of T of finite multiplicity,* $\mathrm{asc}(T - \lambda) < \infty$ *and $\mathcal{R}(T - \lambda)$ is closed.* $\diamondsuit$

Corollary 2.2 [24]. *Let $T \in \mathcal{L}(X)$. Then,*
(i) $\sigma_b(T) = \sigma_e(T) \cup acc\, \sigma(T)$,
(ii) $\sigma_{ub}(T) = \sigma_{e_1}(T) \cup acc\, \sigma_{ap}(T)$,
(iii) $\sigma_{lb}(T) = \sigma_{e_2}(T) \cup acc\, \sigma_{\delta}(T)$,

where $acc\, \sigma(T)$ is the set of accumulation points of $\sigma(T)$. $\diamondsuit$

Now, let us recall the following definition introduced in [21].

Definition 2.5. Let X be a Banach space. $T : \mathcal{D}(T) \longrightarrow X$ is said to be weakly demicompact, if every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $x_n - Tx_n$ converges weakly in X, has a weakly convergent subsequence. $\diamondsuit$

3. Generalized Weak Demicompactness

In the rest of this paper, we denote $\rightarrow$ for the strong convergence and $\rightharpoonup$ for the weak convergence.
Now, we introduce the following definition:

Definition 3.1. Let X be a Banach space. An operator $A \in \mathcal{L}(X)$ is called a generalized weakly demicompact operator if there exists a finite subset E of $\mathbb{C}$ containing 0 such that

(i) For all $\lambda \in \mathbb{C}\backslash E$, $\frac{1}{\lambda}A$ is weakly demicompact operator,
(ii) for all $\lambda \in \mathbb{C}\backslash E$, $A - \lambda$ has a finite ascent, and
(iii) all $\lambda \in \sigma(A)\backslash E$, are eigenvalues of finite multiplicity and have no accumulation point except possibly points of E. $\diamondsuit$

The set E is called a generalized set of A.

Remark 3.1. *It should be noted that if, E is a generalized set of A and G is a finite subset of $\mathbb{C}$ containing E, then G is also a generalized set of A.* $\diamondsuit$

Now, we will show that there is a relationship between the class of generalized weakly demicompact operators and upper semi-Fredholm operators acting on a Banach space. First, let us recall the following definition:

Definition 3.2. An operator $T\colon \mathcal{D}(T) \longrightarrow Y$ is said to be weakly closed, if for every sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $x_n \rightharpoonup x$ in X and $Tx_n \rightharpoonup y$ in Y implies $x \in \mathcal{D}(T)$ and $Tx = y$. $\diamondsuit$

By $\mathcal{C}_W(X)$ we denote the set of all weakly closed, densely defined linear operators from X into itself.

Remark 3.2. *Clearly, every weakly closed operator is closed.* $\diamondsuit$

The following key lemma will be useful for some proofs.

Lemma 3.1. *Let X be a Banach space and $T \in \mathcal{C}(X)$. If T is a demicompact operator, then T is a weakly demicompact operator.* $\diamondsuit$

Proof. Suppose that T is a demicompact operator. Let $(x_n)_n$ be a bounded sequence such that $y_n := x_n - Tx_n \rightharpoonup y$. Since T is a demicompact operator, using Theorem 3.1 in [8], we deduce that $I - T \in \Phi_+(X)$ which implies that $I - \widehat{T} \in \Phi_+(X_T, X)$. The use of $\alpha(I - T) < \infty$, implies that there exists a closed subspace X_0 of X such that

$$X = \mathcal{N}(I - T) \oplus X_0.$$

Since X_0 is a closed subset of X and $T \in \mathcal{C}(X)$, we deduce that $X_0 \cap \mathcal{D}(T)$ is a closed subset of X_T. Therefore $(X_0 \cap \mathcal{D}(T), \|.\|_T)$ is a Banach space. Hence,

$$X_T = \mathcal{N}(I - \widehat{T}) \oplus [\mathcal{D}(T) \cap X_0].$$

Then, $(I - \widehat{T})_{|X_0 \cap \mathcal{D}(T)} : (X_0 \cap \mathcal{D}(T), \|.\|_T) \longrightarrow (\mathcal{R}(I - T), \|.\|)$ is invertible with bounded inverse on $\mathcal{R}(I - T) = \mathcal{R}(I - \widehat{T})$. Then from the boundedness of $((I - \widehat{T})_{|X_0 \cap \mathcal{D}(T)})^{-1}$ on $\mathcal{R}(I - \widehat{T})$, we have $x_n \rightharpoonup (I - \widehat{T})^{-1}(y)$. Therefore, by applying Proposition 3.5 [7], we get $(I - \widehat{T})x_n \to y$, so $(I - T)x_n \to y$, from the demicompactness of T, there exists a convergent subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that $x_{\varphi(n)} \to x$. Hence, $x_{\varphi(n)} \rightharpoonup x$ and this achieves the proof. $\square$

Now, we state the following result:

Theorem 3.1. *Let X be a Banach space and $T \in \mathcal{C}_W(X)$. T is a generalized weakly demicompact if, and only if, there exists a finite subset $E \subset \mathbb{C}$ containing* 0 *such that $\lambda - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C} \backslash E$.* $\diamondsuit$

Proof. Suppose that, there exists a finite subset $E \subset \mathbb{C}$ containing 0 such that $\lambda - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C} \backslash E$, then T is a generalized Riesz operator. Thus, it follows that, $\operatorname{asc}(\lambda - T) < \infty$ for all $\lambda \in \mathbb{C} \backslash E$ and all $\lambda \in \sigma(T) \backslash E$ are eigenvalues of finite multiplicity and have no accumulation point except possibly points of E. On the other hand, from Theorem 2.1 [19], $\lambda - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C} \backslash E$ implies that $\frac{1}{\lambda}T$ is a demicompact operator for all $\lambda \in \mathbb{C} \backslash E$, then from Lemma 3.1, we infer that $\frac{1}{\lambda}T$ is a weakly demicompact operator for all $\lambda \in \mathbb{C}\backslash E$. Thus, we conclude that T is a generalized weakly demicompact operator.

To prove the converse, let us first check that $\alpha(\lambda - T) < \infty$. Since T is a generalized weakly demicompact operator, it follows that $\operatorname{asc}(\lambda - T) < \infty$ for all $\lambda \in \mathbb{C}\backslash E$. We denote by $p = \operatorname{asc}(\lambda - T) < \infty$. Furthermore, for any eigenvalue $\lambda \in \sigma(T)\backslash E$, we have $\operatorname{mult}(T, \lambda) < \infty$ which implies that

$$\operatorname{mult}(T, \lambda) = \dim \mathcal{N}^{\infty}(\lambda - T) = \dim \mathcal{N}((\lambda - T)^p) < \infty.$$

Hence, from the following inclusion

$$\mathcal{N}(\lambda - T) \subset \mathcal{N}((\lambda - T)^p),$$

we get $\dim \mathcal{N}(\lambda - T) = \alpha(\lambda - T) < \infty$.

For the second part of this proof, we claim that $\mathcal{R}(\lambda - T)$ is closed. Let $(x_n)_n$ be any sequence included in $\mathcal{D}(T)$. Let us take $y_n = (\lambda - T)x_n$ such that $y_n \to y \in X$. Since $\frac{1}{\lambda}T$ is a weakly demicompact and

$$x_n - \frac{1}{\lambda}Tx_n \rightharpoonup \frac{1}{\lambda}y \in X,$$

there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that $x_{\varphi(n)} \rightharpoonup x \in X$. Using both

$$x_{\varphi(n)} \rightharpoonup x \text{ and } (\lambda - T)x_{\varphi(n)} \rightharpoonup y,$$

together with the closedness of $(\lambda - T)$, we infer that

$$x \in \mathcal{D}(T) \text{ and } y = (\lambda - T)x.$$

As result, $y \in \mathcal{R}(\lambda - T)$. Hence, $\mathcal{R}(\lambda - T)$ is closed. □

The following result will show the connection between Fredholm operators having null indices, and the class of generalized weakly demicompact operators.

Theorem 3.2. *Let X be a Banach space and $T \in \mathcal{C}_W(X)$. If μT is a generalized weakly demicompact operator for each $\mu \in [0, 1]$ with a generalized subset E, then $\lambda - T \in \Phi(X)$ and $i(\lambda - T) = 0$, for all $\lambda \in \mathbb{C}\backslash E$.* ♢

Proof. Since μT is a generalized weakly demicompact operator for each $\mu \in [0, 1]$, then $\lambda - \mu T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$. Now, since the index $i(\lambda - \mu T)$ is continuous in μ and since it is an integer, including infinite values, it must be constant for every $\mu \in [0, 1]$. Showing that $i(\lambda - \mu T) = i(\lambda - T) = i(\lambda) = 0$. Consequently, $\lambda - T \in \Phi(X)$ and $i(\lambda - T) = 0$, for all $\lambda \in \mathbb{C}\backslash E$. □

Now, we give a characterization of generalized weakly demicompact operators by means of a measure of weak noncompactness.

Theorem 3.3. *Let X be a Banach space and $T \in \mathcal{C}_W(X)$ such that $T(\mathcal{D}(T)) \subset \mathcal{D}(T)$. Let μ be a measure of weak noncompactness and let Ψ_μ be the associated measure. If $\widehat{T}^m$ is a Dunford–Pettis operator for some $m \in \mathbb{N}\backslash\{0\}$ and $\lim_{n\to 0} \Psi_\mu((\widehat{T}^n))^{\frac{1}{n}} = 0$, then T is a generalized weakly demicompact.* ♢

Proof. Let $\lambda \in \mathbb{C}\backslash\{0\}$. By applying Theorem 1 in [31], to show that $\lambda - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$, it suffices to prove that, for any $K \in \mathcal{K}(X)$, $\alpha(\lambda - T - K) < \infty$. To do so, it suffices to establish that, the set $F = \mathcal{N}(\lambda - T - K) \cap \overline{B}_X$ is compact.
Since, $\lim_{n\to 0} \Psi_\mu((\widehat{T}^n))^{\frac{1}{n}} = 0$, there exists $n_0 \geq m$ such that for all $n \geq n_0$,

$$\Psi_\mu((\widehat{T}^{n_0}))^{\frac{1}{n_0}} < |\lambda|. \tag{3.1}$$

Also, we can write, for $n \geq n_0$,

$$\lambda^{n_0} - \widehat{T}^{n_0} = \sum_{j=0}^{n_0-1} \lambda^j \widehat{T}^{n_0-1-j}(\lambda - \widehat{T}).$$

Let $x \in F$, then

$$\sum_{j=0}^{n_0-1} \lambda^j \widehat{T}^{n_0-1-j}(\lambda - \widehat{T})(x) = \sum_{j=0}^{n_0-1} \lambda^j \widehat{T}^{n_0-1-j} K(x).$$

Which implies that

$$\lambda^{n_0} x = \widehat{T}^{n_0}(x) + \sum_{j=0}^{n_0-1} \lambda^j \widehat{T}^{n_0-1-j} K(x),$$

and it follows that

$$x = \frac{1}{\lambda^{n_0}} \widehat{T}^{n_0}(x) + \frac{1}{\lambda^{n_0}} \sum_{j=0}^{n_0-1} \lambda^j \widehat{T}^{n_0-1-j} K(x).$$

Therefore,

$$F \subset \frac{1}{\lambda^{n_0}} \widehat{T}^{n_0}(F) + \frac{1}{\lambda^{n_0}} \sum_{j=0}^{n_0-1} \lambda^j \widehat{T}^{n_0-1-j} K(F). \tag{3.2}$$

Let $\Omega := \frac{1}{\lambda^{n_0}} \widehat{T}^{n_0}(F) + \frac{1}{\lambda^{n_0}} \sum_{j=0}^{n_0-1} \lambda^j \widehat{T}^{n_0-1-j} K(F)$.

Thus,

$$\mu(F) \leq \frac{1}{|\lambda|^{n_0}} \Psi_\mu(\widehat{T}^{n_0})\mu(F) + \Psi_\mu\left(\frac{1}{\lambda^{n_0}} \sum_{j=0}^{n_0-1} \lambda^j \widehat{T}^{n_0-1-j}\right) \Psi_\mu(K)\mu(F).$$

Moreover, K is a weakly compact operator, then $\Psi_\mu(K) = 0$ and using Eq. (3.1), we infer that $\mu(F) < \mu(F)$, which is absurd. Therefore, $\mu(F) = 0$ and consequently F is a relatively weakly compact including in Ω. Taking into account that $\widehat{T}^{n_0}$ is a Dunford–Pettis operator and $F \subset \mathcal{D}(T)$, then $\widehat{T}^{n_0}(F) = T^{n_0}(F)$ is compact. From Eq. (3.2), it follows that F is compact and so $\lambda - T \in \Phi_+(X)$, for all $\lambda \in \mathbb{C}\backslash\{0\}$. Finally, applying Theorem 3.1 with $E = \{0\}$, we deduce that T is a generalized weakly demicompact operator. □

Theorem 3.4. *Let X be a Banach space and $A \in \mathcal{L}(X)$ such that $0 \in \sigma_{ub}(A)$. Then, A is a generalized weakly demicompact if, and only if, A has a finite upper semi-Browder spectrum.* ◇

Proof. Suppose that A is a generalized weakly demicompact. Then, from Theorem 3.1, there exists a finite set $E \subset \mathbb{C}$ containing 0 such that for all $\lambda \in \mathbb{C}\backslash E$, $A-\lambda \in \Phi_+(X)$, with finite ascent, therefore $\sigma_{ub}(A) \subset E$ and hence $\sigma_{ub}(A)$ is a finite subset.
To prove the converse, suppose that $\sigma_{ub}(A)$ is a finite subset of $\mathbb{C}$. For all $\lambda \in \mathbb{C}\backslash\sigma_{ub}(A)$, $A-\lambda$ is an upper semi-Browder, then $A-\lambda$ is an upper semi-Fredholm which implies that $\frac{1}{\lambda}A$ is a demicompact for all $\lambda \in \mathbb{C}\backslash\sigma_{ub}(A)$, so from Lemma 3.1, we infer that $\frac{1}{\lambda}A$ is a weakly demicompact for all $\lambda \in \mathbb{C}\backslash\sigma_{ub}(A)$. Furthermore, we have $\mathrm{asc}(A-\lambda) = p < \infty$, for all $\lambda \in \mathbb{C}\backslash\sigma_{ub}(A)$. Let $\lambda \in \sigma(A)\backslash\sigma_{ub}(A)$. Since $\lambda \in \sigma(A)$, using Corollary 2.2, λ is an isolated point of $\sigma_{ap}(A)$. Hence, $\lambda \in \sigma_{ap}(A)\backslash\sigma_{ub}(A)$. Now, applying Corollary 2.1, we deduce that λ is an eigenvalue of A of finite multiplicity. Again, using Corollary 2.2, we get all $\lambda \in \sigma(A)\backslash\sigma_{ub}(A)$ have no accumulation point except possibly points of $\sigma_{ub}(A)$. Therefore, A is a generalized weakly demicompact with a generalized set $\sigma_{ub}(A)$. $\square$

Theorem 3.5. *Let X be a Banach space, $A \in \mathcal{L}(X)$ and P be a non-zero complex polynomial, with $z_1 \cdots z_p$ its zeros, such that $P(A)$ is a generalized weakly demicompact with a generalized set $E = \{0\}$. Then, A is a generalized weakly demicompact with a generalized set $E^{'} = \{0, z_1 \cdots z_p\}$.* $\diamondsuit$

Proof. Suppose that there exists P a non-zero complex polynomial such that $P(A)$ is a generalized weakly demicompact with a generalized set $E = \{0\}$. Put $P(z) = \sum_{k=0}^{p} a_k z^k$. Hence for $\omega \in \mathbb{C}\backslash\{0, z_1 \cdots z_p\}$, we have the following equality:

$$P(\omega) - P(A) = \sum_{k=1}^{p} a_k(\omega^k - A^k).$$

On the other hand, for any $k \in \{1, \dots, p\}$

$$\omega^k - A^k = (\omega - A)\sum_{i=0}^{k-1} \omega^i A^{k-1-i}.$$

This allows us to write

$$P(\omega) - P(A) = (\omega - A)Q(A) = Q(A)(\omega - A), \tag{3.3}$$

where

$$Q(A) = \sum_{k=1}^{p} a_k \sum_{i=0}^{k-1} \omega^i A^{k-1-i}.$$

Let $n \in \mathbb{N}$, using Eq. (3.3), we have

$$(P(\omega) - P(A))^n = (\omega - A)^n[Q(A)]^n = [Q(A)]^n(\omega - A)^n.$$

Hence,

$$\mathcal{N}\left((\omega - A)^n\right) \subset \mathcal{N}\left((P(\omega) - P(A))^n\right), \text{ for all } n \in \mathbb{N}.$$

This implies

$$\bigcup_{n\in\mathbb{N}} \mathcal{N}\left((\omega - A)^n\right) \subset \bigcup_{n\in\mathbb{N}} \mathcal{N}\left((P(\omega) - P(A))^n\right). \tag{3.4}$$

From Definition 3.1, we have for all $\mu \in \mathbb{C}\backslash\{0\}$, $\operatorname{asc}(\mu - P(A)) < \infty$. Then, in particular, for $\mu = P(\omega) \neq 0$, we get $\operatorname{asc}(P(\omega) - P(A)) < \infty$, suppose that $\operatorname{asc}(P(\omega) - P(A)) = n_0$, then

$$\dim \bigcup_{n\in\mathbb{N}} \mathcal{N}\left((P(\omega) - P(A))^n\right) = \dim \mathcal{N}\left((P(\omega) - P(A))^{n_0}\right) < \infty.$$

Using Equation (3.4), we have

$$\dim \bigcup_{n\in\mathbb{N}} \mathcal{N}\left((\omega - A)^n\right) < \infty,$$

this implies that $\operatorname{asc}(\omega - A) < \infty$ and we have also $\alpha(\omega - A) < \infty$. Now, we have to show that $\mathcal{R}(\omega - A)$ is closed. The use of $\alpha(\omega - A) < \infty$, implies that there exists a closed subspace X_0 of X such that

$$X = \mathcal{N}(\omega - A) \oplus X_0.$$

In view of Theorem 3.14 in [27], it suffices to prove that there is a constant $c > 0$ such that for every $x \in X_0$, $\|(\omega - A)x\| \geq c\|x\|$. If not, there exists a sequence $(x_n)_n$ of X_0 such that $\|(\omega - A)x_n\| \to 0$ and $\|x_n\| = 1$. Again, since $\alpha(P(\omega) - P(A)) < \infty$, then there exists a closed subspace X_1 of X such that

$$X = \mathcal{N}\left(P(\omega) - P(A)\right) \oplus X_1.$$

Then, $(P(\omega) - P(A))_{|X_1} : X_1 \longrightarrow \mathcal{R}\left(P(\omega) - P(A)\right)$ is invertible with bounded inverse on $\mathcal{R}\left(P(\omega) - P(A)\right)$. Also, we have

$$(\omega - A)x_n \to 0. \tag{3.5}$$

Thus, using Eqs. (3.3) and (3.5), we have

$$\left(P(\omega) - P(A)\right)x_n \to 0.$$

From the boundedness of $\left[(P(\omega) - P(A))_{|X_1}\right]^{-1}$ on $\mathcal{R}\left(P(\omega) - P(A)\right)$, we deduce that $x_n \to 0$. This contradicts the continuity of the norm. So, we conclude that $(\omega - A) \in \Phi_+(X)$, for all $\omega \in \mathbb{C}\backslash\{0, z_1 \cdots z_p\}$.
Hence, from Theorem 3.1, A is a generalized weakly demicompact with a generalized set $E^{'} = \{0, z_1 \cdots z_p\}$. □

Proposition 3.1. *Let X be a Banach space and $A \in \mathcal{L}(X)$. Assume that $\Omega \neq \emptyset$ is a connected open subset of $\mathbb{C}$ such that $\sigma(A) \subset \Omega$ and let $f : \Omega \longrightarrow \mathbb{C}$ be an analytic function. If $f(A) \in \mathcal{R}(X)$, then A is a generalized weakly demicompact.* ♢

Proof. If $f^{-1}\{0\} \subset \sigma(A)$, f being analytic in the compact $\sigma(A)$, then they are in finite number $\{\lambda_1, \cdots, \lambda_n\}$, so we can write

$$f(z) = \prod_{i=1}^{n}(z - \lambda_i)^{\alpha_i} g(z),$$

where α_i denotes the multiplicity of λ_i and g is an analytic function on neighborhood of $\sigma(A)$ such that $g(z)$ has no zero in $\sigma(A)$. Let $h(z) = \frac{1}{g(z)}$, h is an analytic function on neighborhood of $\sigma(A)$. Observe that

$$\prod_{i=1}^{n}(z-\lambda_i)^{\alpha_i} = f(z)h(z) = h(z)f(z),$$

then from Lemma 6.15 [27],

$$\prod_{i=1}^{n}(A-\lambda_i)^{\alpha_i} = f(A)h(A) = h(A)f(A).$$

Since $f(A) \in \mathcal{R}(X)$ and f, h commutes, we deduce that $\prod_{i=1}^{n}(A-\lambda_i)^{\alpha_i} \in \mathcal{R}(X)$, which implies that $\prod_{i=1}^{n}(A-\lambda_i)^{\alpha_i}$ is a generalized weakly demicompact operator with $E = \{0\}$. Therefore, from Theorem 3.5, A is a generalized weakly demicompact operator with a generalized set $E^{'} = \{0, \lambda_1, \cdots, \lambda_n\}$. $\square$

Theorem 3.6. *Let T, S be two bounded commuting linear operators on a Banach space X, and suppose that there exists a non-zero complex polynomial P such that $P(T) \in \mathcal{R}(X)$. Let m_T be its minimal polynomial. If S is a generalized weakly demicompact with a generalized set E, and $m_T(\lambda) \neq 0$ for all $\lambda \in E$, then $S - T \in \Phi_+(X)$.* $\diamondsuit$

Proof. Let $m_T(z) = \prod_{i=1}^{n}(z-\lambda_i)$ be the minimal polynomial of T. Since S is a generalized weakly demicompact operator, then from Theorem 3.1, we get $S - \lambda \in \Phi_+(X)$, for all $\lambda \in \mathbb{C}\backslash E$. Therefore, since $m_T(\lambda) \neq 0$ for all $\lambda \in E$, then

$$m_T(S) = \prod_{i=1}^{n}(S-\lambda_i), \text{ with } \lambda_i \in \mathbb{C}\backslash E.$$

For all $i = 1, \ldots, n$, $S - \lambda_i \in \Phi_+(X)$, hence by [27], we get $m_T(S) \in \Phi_+(X)$. Since $m_T(T) \in \mathcal{R}(X)$, furthermore, $m_T(T)$ and $m_T(S)$ commutes, it follows from Lemma 2.1 that, $m_T(S) - m_T(T) \in \Phi_+(X)$. We can write

$$m_T(S) - m_T(T) = Q(S,T)(S-T),$$

where Q is a polynomial of S and T. Therefore, $Q(S,T)(S-T) \in \Phi_+(X)$. Consequently, Theorem 5.32 [27] yields that, $S-T$ is an upper semi-Fredholm operator. $\square$

Before moving to the next section, we give examples of generalized weakly demicompact operators:

Examples 3.1. (i) Obviously, Riesz operators are generalized weakly demicompact with a generalized set $E = \{0\}$.

(ii) Let C_w be the space of continuous w-periodic functions $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ and C_w' the space of continuously differentiable w-periodic functions $x : \mathbb{R} \longrightarrow \mathbb{R}^n$. C_w equipped with the maximum norm $\|\cdot\|_\infty$ and C_w'

with the norm given by $\|\cdot\|_\infty^1 = \max\{\|u\|_\infty, \|u'\|_\infty\}$ for $u \in C'_w$ are Banach spaces. Let us consider the following differential equation:

$$x'(t) = a(t)x'(t-h_1) + b(t)x(t-h_2) + f(t).$$

Here, a and b are continuous w-periodic functions such that $|a(t)| < k$, $(k < 1)$, where $k < \frac{1}{w}$ if $w > 2$ or $k < \frac{1}{2}$ if $w \leq 2$; $f \in C_w$ is a given function and $x \in C'_w$ is an unknown function. Observe that the last equation can be rewritten in the operator form

$$Bx - Ax = f,$$

where $B : C'_w \longrightarrow C_w$ is given by the formula:

$$(Bx)(t) = x'(t),$$

and the operator $A : C'_w \longrightarrow C_w$ by the formula:

$$(Ax)(t) = a(t)x'(t-h_1) + b(t)x(t-h_2).$$

Let us consider the polynomial $P(X) = \frac{1}{2}X^2$ and the operator $T = {c^{D^{(\frac{1}{2})}}}$; where $c^{D^{(\frac{1}{2})}}$ is the Caputo derivative of fractional order $\frac{1}{2}$. T is a Dunford–Pettis operator. Indeed, first write

$$Tx(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} x'(\tau)\mathrm{d}\tau,$$

and let us consider the set $U := \{x \in C^1[0,w]$ such that $\|x'\|_\infty \leq k\}$.
Now, for $0 \leq t_1 \leq t_2 \leq w$, we can write

$$\begin{aligned}
|Tx(t_1) - Tx(t_2)| &= \frac{1}{\Gamma(\frac{1}{2})}\left|\int_0^{t_1}(t_1-\tau)^{-\frac{1}{2}}x'(\tau)\mathrm{d}\tau - \int_0^{t_2}(t_2-\tau)^{-\frac{1}{2}}x'(\tau)\mathrm{d}\tau\right| \\
&\leq \frac{\|x'\|_\infty}{\Gamma(\frac{1}{2})}\left|\int_0^{t_1}\left[(t_1-\tau)^{-\frac{1}{2}} - (t_2-\tau)^{-\frac{1}{2}}\right]\mathrm{d}\tau \right.\\
&\quad \left. + \int_{t_1}^{t_2}(t_2-\tau)^{-\frac{1}{2}}\mathrm{d}\tau\right| \\
&= \frac{\|x'\|_\infty}{\Gamma(\frac{3}{2})}\left(2(t_2-t_1)^{\frac{1}{2}} + t_1^{\frac{1}{2}} - t_2^{\frac{1}{2}}\right) \\
&\leq \frac{\|x'\|_\infty}{\Gamma(\frac{3}{2})}\left(2(t_2-t_1)^{\frac{1}{2}}\right). \qquad (3.6)
\end{aligned}$$

This proves that Tx is a continuous function.
Now, it remains to show that $T(U) := \{Tx,\ x \in U\}$ is a relatively compact set. Note that, for $z \in T(U)$ and $t \in [0,w]$, we have the following estimate:

$$|z(t)| \leq \frac{1}{\Gamma(\frac{3}{2})}\|x'\|_\infty w^{\frac{1}{2}},$$

which is the required boundedness. Moreover, for $0 \leq t_1 \leq t_2 \leq w$, it follows from Eq. (3.6), that

$$|Tx(t_1) - Tx(t_2)| \leq \frac{2\|x'\|_\infty}{\Gamma(\frac{3}{2})}(t_2-t_1)^{\frac{1}{2}}.$$

Thus, if $|t_2 - t_1| < \eta$, then

$$|Tx(t_1) - Tx(t_2)| \leq \frac{2\|x'\|_\infty}{\Gamma(\frac{3}{2})}\eta^{\frac{1}{2}} \leq \frac{2k}{\Gamma(\frac{3}{2})}\eta^{\frac{1}{2}}.$$

We see that the set $T(U)$ is equicontinuous. Then, the Arzela–Ascoli Theorem yields that $T(U)$ is relatively compact.
Now, when applying Theorem 3.3 and Theorem 3.2 [22] we obtain

$$P(T)(x) = \frac{1}{2}T^2(x) = \frac{1}{2}\left[c^{D^{(\frac{1}{2})}}\right]^2 x(t) = \frac{1}{2}x'(t).$$

Clearly, $P(T)$ is a bounded operator and $\|P(T)\| \leq \dfrac{1}{2}$ and so, $P(T)$ is a weakly sequentially continuous (see Definition 1.3.3 in [18]). Thus, we get

$$\mu\left(P(T)(B_{C_w})\right) \leq \frac{1}{2}\mu(B_{C'_w}).$$

Since C'_w is not reflexive, we obtain that

$$\Psi_\mu\left(P(T)\right) \leq \frac{1}{2} < 1.$$

Hence, from Theorem 3.3, we deduce that $P(T)$ is a generalized weakly demicompact operator and thus, Theorem 3.5 shows that T is a generalized weakly demicompact operator.

4. Generalized Weak Demicompactness Results for Matrix Operator

In this section we will use some conditions introduced in [28], imposed on the entries of the operator L_0 to ensure its closedness. Having obtained the closure of the operator L_0, we will investigate after its essential spectra. For this we consider, in the product space $X \times X$, the following matrix operator

$$L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.1}$$

where the operator A acts on X and has domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space X, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp., $\mathcal{D}(C)$) and acts on X. Now, let us consider the following assumptions [28].

(H1) A is closed, densely defined linear operator on X with nonempty resolvent set $\rho(A)$.

(H2) The operator B is a densely defined linear operator on X and for some (hence all) $\mu \in \rho(A)$, the operator $(A-\mu)^{-1}B$ is closable. (In particular, if B is closable, then $(A-\mu)^{-1}B$ is closable).

(H3) The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence all) $\mu \in \rho(A)$, the operator $C(A-\mu)^{-1}$ is bounded. (In particular, if C is closable, then $C(A-\mu)^{-1}$ is bounded).

(H4) $\mathcal{D}(B) \subset \mathcal{D}(D)$.

(H5) For some (hence all) $\mu \in \rho(A)$, the operator $D - C(A-\mu)^{-1}B$ is closable. We will denote by $S(\mu)$ its closure.

Remark 4.1. (i) *Under the assumptions* (H1) *and* (H2), *we infer that for each* $\mu \in \rho(A)$ *the operator* $G(\mu) := \overline{(A-\mu)^{-1}B}$ *is bounded on* X.

(ii) *From the assumption* (H3), *it follows that the operator:* $F(\mu) := C(A-\mu)^{-1}$ *is bounded on* X. ♢

We recall the following result which describes the closure of L_0.

Theorem 4.1 [3]. *Let conditions* (H1)–(H3) *be satisfied and the lineal* $\mathcal{D}(B) \cap \mathcal{D}(D)$ *be dense in* X. *Then the operator* L_0 *is closable and the closure* L *of* L_0 *is given by:*

$$L = \mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - A & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}. \tag{4.2}$$

♢

Or, spelled out,

$$L : \mathcal{D}(L) \subset (X \times X) \longrightarrow X \times X$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(x + G(\mu)y) - \mu G(\mu)y \\ C(x + G(\mu)y) - S(\mu)y \end{pmatrix},$$

with $\mathcal{D}(L) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times X \text{ such that } x + G(\mu)y \in \mathcal{D}(A) \text{ and} y \in \mathcal{D}(S(\mu)) \right\}$. Note that the description of the operator L does not depend on the choice of the point $\mu \in \rho(A)$.

Remark 4.2. *Let* $\lambda \in \mathbb{C}$. *It follows from Eq.* (4.2) *that*

$$\begin{aligned} \lambda - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \mu)M(\mu) \\ &:= UV(\lambda)W - (\lambda - \mu)M(\mu), \end{aligned}$$

where $M(\mu) = \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}$. ♢

We start this section by a result for a particular representation of L_0.

Lemma 4.1. *Let* $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(X)$ *and consider the* 2×2 *operator matrix*

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in \mathcal{L}(X)$. *Then,* A *and* B *are generalized weakly demicompact operators if, and only if,* M_C *is a generalized weakly demicompact operator.* ♢

Proof. Let $\lambda \in \mathbb{C}$. Clearly, we have

$$\lambda - M_C = \begin{pmatrix} \lambda - A & -C \\ 0 & \lambda - B \end{pmatrix}.$$

Since A and B are generalized weakly demicompact, then there exists two finite subsets E_1 and E_2 of $\mathbb{C}$ containing 0 such that $\lambda - A \in \Phi_+(X)$ for all

$\lambda \in \mathbb{C}\backslash E_1$ and $\lambda - B \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E_2$. From Remark 3.1, it follows that A and B are also generalized weakly demicompact with the generalized set $E = E_1 \cup E_2$. This allows us to get $\lambda - A \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$ and $\lambda - B \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E$. Now, when applying Lemma 6.6.1 [16], we get $\lambda - M_C \in \Phi_+(X \times X)$ for all $\lambda \in \mathbb{C}\backslash E$, and for every $C \in \mathcal{L}(X)$. Hence, from Theorem 3.1, M_C is a generalized weakly demicompact with the generalized set E.
To prove the converse, assume that M_C is a generalized weakly demicompact operator then, from Theorem 3.1, there exists a finite subset E of $\mathbb{C}$ contains 0 such that $\lambda - M_C$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C}\backslash(E \cap \sigma(B))$.
First, we put the following factorization

$$\lambda - M_C = \begin{pmatrix} \lambda - A & -C \\ 0 & \lambda - B \end{pmatrix} = \begin{pmatrix} I & -C(\lambda - B)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \lambda - B \end{pmatrix}.$$

Since we have found that

$$\begin{pmatrix} \lambda - A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \lambda - B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \lambda - B \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & I \end{pmatrix},$$

and using the fact that $\lambda - M_C \in \Phi_+(X \times X)$, it follows from Lemma 6.6.2 [16], that $\lambda - A$ and $\lambda - B$ are upper semi-Fredholm, for all $\lambda \in \mathbb{C}\backslash E$. Thus, Theorem 3.1 yields that A and B are generalized weakly demicompact operators. □

Now, let us introduce the following set:

$$\Lambda_X^G = \{J \in \mathcal{L}(X) \text{ such that } \theta J \text{ is generalized weakly demicompact for every } \theta \in [0,1]\},$$

and rewrite the Frobenius–Schur factorization:

$$\alpha L = \alpha\mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \alpha\mu - \alpha A & 0 \\ 0 & \alpha\mu - \alpha S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}.$$

Let $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} \lambda - \alpha L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda - \alpha A & 0 \\ 0 & \lambda - \alpha S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \alpha\mu)M(\mu) \\ &:= UV_\alpha(\lambda)W - (\lambda - \alpha\mu)M(\mu). \end{aligned} \tag{4.3}$$

We are now in the position to express the main result of this section.

Theorem 4.2. *Let X be a Banach space. Assume that the operator L_0 defined in Eq. (4.1) satisfies (H_1)-(H_5). Let $\mu \in \rho(A)$ and $\lambda \in \mathbb{C}$. If for $\alpha \in [0,1]$ such that $\lambda \neq \alpha\mu$, the operators αA and $\alpha S(\mu)$ are generalized weakly demicompact, there exists $T_{\lambda l}^{\alpha}$ a left Fredholm inverse of $UV_\alpha(\lambda)W$ such that $M(\mu)T_{\lambda l}^{\alpha} \in \Lambda_X^G$, then αL is a generalized weakly demicompact for all $\alpha \in [0,1]$.* ♢

Proof. We assume that the assumption holds. Let $\mu \in \rho(A)$ and $\lambda \in \mathbb{C}$. According to Eq. (4.3), we have

$$\lambda - \alpha L = UV_\alpha(\lambda)W - (\lambda - \alpha\mu)M(\mu).$$

Let $T^{\alpha}_{\lambda} := UV_{\alpha}(\lambda)W$. Since αA and $\alpha S(\mu)$ are generalized weakly demicompact, from Lemma 4.1, we deduce that there exists a finite subset E containing 0 such that $V_{\alpha}(\lambda)$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C}\backslash E$. Let $E_1 := E \cup E^{-1}$, where the set $E^{-1} := \{\frac{1}{\alpha},\ \alpha \in E\backslash\{0\}\}$. Clearly, $V_{\alpha}(\lambda)$ is an upper semi-Fredholm operator, for all $\lambda \in \mathbb{C}\backslash E_1$. Hence, there exists a left Fredholm inverse $V^{\alpha}_{\lambda l}$ of $V_{\alpha}(\lambda)$ and according to Proposition 6.7.1 [16], it is defined as follows:

$$V^{\alpha}_{\lambda l} = \begin{pmatrix} A^{\alpha}_{\lambda l} & K_1 \\ K_2 & S^{\alpha}_{\lambda l}(\mu) \end{pmatrix},$$

where $A^{\alpha}_{\lambda l}$, $S^{\alpha}_{\lambda l}(\mu)$ are left Fredholm inverse, respectively, of $\lambda - \alpha A$ and $\lambda - \alpha S(\mu)$ and K_1, K_2 are compact operators. Now, using Lemma 6.7.1 [16], we deduce that $T^{\alpha}_{\lambda l} := W^{-1}V^{\alpha}_{\lambda l}U^{-1}$ is a left Fredholm inverse of T^{α}_{λ}, then there exists $K \in \mathcal{K}(X \times X)$ such that

$$T^{\alpha}_{\lambda l}T^{\alpha}_{\lambda} = I - K.$$

Thus we can write

$$\begin{aligned} \lambda - \alpha L &= T^{\alpha}_{\lambda} - (\lambda - \alpha\mu)M(\mu) \\ &= T^{\alpha}_{\lambda} - (\lambda - \alpha\mu)M(\mu)\left[T^{\alpha}_{\lambda l}T^{\alpha}_{\lambda} + K\right] \\ &= \left[I - (\lambda - \alpha\mu)M(\mu)T^{\alpha}_{\lambda l}\right]T^{\alpha}_{\lambda} - (\lambda - \alpha\mu)M(\mu)K. \qquad (4.4) \end{aligned}$$

As $M(\mu)T^{\alpha}_{\lambda l} \in \Lambda^{G}_{X}$, it follows from Theorem 3.1 that $(\beta - \theta M(\mu)T^{\alpha}_{\lambda l}) \in \Phi_{+}(X \times X)$ for all $\beta \in \mathbb{C}\backslash E_1$. Moreover, the fact that $\mu \in \rho(A)$ implies that $\alpha(\mu - A) \in \Phi(X)$, so $\alpha\mu \in \rho(A)\backslash E_1$, hence we get $\frac{1}{\lambda - \alpha\mu} \in \mathbb{C}\backslash E_1$. Therefore, $I - (\lambda - \alpha\mu)M(\mu)T^{\alpha}_{\lambda l} \in \Phi_{+}(X \times X)$. Now, when applying Theorem 5.26 [27] on Eq. (4.4), we deduce that the first product term is an upper semi-Fredholm. Since $(\lambda - \alpha\mu)M(\mu)K \in \mathcal{K}(X \times X) \subset \mathcal{F}_{+}(X \times X)$, it follows that $\lambda - \alpha L \in \Phi_{+}(X \times X)$ for all $\lambda \in \mathbb{C}\backslash E_1$. Hence, from Theorem 3.1, αL is a generalized weakly demicompact operator with the generalized set E_1. □

Corollary 4.1. *Assume that the operator L_0 defined in Eq. (4.1) satisfies* (H_1)–(H_5). *Let $\mu \in \rho(A)$ and $\lambda \in \mathbb{C}$ such that $\mu \neq \lambda$. If A and $S(\mu)$ are generalized weakly demicompact operators, there exists $T_{\lambda l}$ a left Fredholm inverse of $UV(\lambda)W$ such that $M(\mu)T_{\lambda l} \in \Lambda^{G}_{X}$, then L is a generalized weakly demicompact operator.* ♢

Proof. The proof is a direct application of Theorem 4.2 for $\alpha = 1$. □

Proposition 4.1. *Let X be a Banach space. Assume that the operator L_0 defined in Eq. (4.1) satisfies (H_1)–(H_5). Let $\lambda \in \mathbb{C}$, if D and A are generalized weakly demicompact and $B \in \mathcal{F}_{+}(X)$, then L is a generalized weakly demicompact operator.* ♢

Proof. We assume that the assumption holds. According to the Frobenius–Schur factorization, we have

$$\begin{aligned} L &= \lambda - \begin{pmatrix} I & 0 \\ F(\lambda) & I \end{pmatrix}\begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - S(\lambda) \end{pmatrix}\begin{pmatrix} I & G(\lambda) \\ 0 & I \end{pmatrix} \\ &:= \lambda - PQ(\lambda)R, \end{aligned}$$

where $F(\lambda) = C(A-\lambda)^{-1}$, $S(\lambda) = \overline{D - C(A-\lambda)^{-1}B}$ and $G(\lambda) = \overline{(A-\lambda)^{-1}B}$. D is a generalized weakly demicompact, then there exists a finite subset $E_1 \subset \mathbb{C}$ containing 0 such that $\lambda - D \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E_1$, which is equivalent to get $\widehat{\lambda - D} \in \Phi_+(X_B, X)$ for all $\lambda \in \mathbb{C}\backslash E_1$. Let $T := C(A-\lambda)^{-1}B$, $Q := \lambda - D$ and $R := \lambda - S(\lambda)$, since $\widehat{B} \in \mathcal{F}_+(X_B, X)$ and $C(A-\lambda)^{-1}$ is bounded, it follows that $\widehat{T} \in \mathcal{F}_+(X_B, X)$. Thus, we obtain for all $\lambda \in \mathbb{C}\backslash E_1$, $\widehat{Q} - \widehat{T} \in \Phi_+(X_B, X)$. Hence, we have $\widehat{Q - T} \in \Phi_+(X_{S(\lambda)}, X)$. As consequence, $\widehat{R} \in \Phi_+(X_{S(\lambda)}, X)$, for all $\lambda \in \mathbb{C}\backslash E_1$, this allows us to deduce that $\lambda - S(\lambda) \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E_1$. On the other hand, since A is a generalized weakly demicompact, it follows that there exists a finite subset E_2 of $\mathbb{C}$ containing 0 such that $\lambda - A \in \Phi_+(X)$ for all $\lambda \in \mathbb{C}\backslash E_2$. Therefore, from Lemma 4.1, we get $Q(\lambda) \in \Phi_+(X \times X)$, for all $\lambda \in \mathbb{C}\backslash E$, where $E = E_1 \cup E_2$. Moreover, the boundedness of the operators P and R gives us that $\lambda - L$ is an upper semi-Fredholm operator for all $\lambda \in \mathbb{C}\backslash E$. Hence, from Theorem 3.1, L is a generalized weakly demicompact. $\square$

In this last part, we will give the relationship between the essential spectrum of the matrix operator L and the essential spectrum of his entries. First, let us recall the following decomposition

$$\begin{aligned}\lambda - \alpha L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda - \alpha A & 0 \\ 0 & \lambda - \alpha S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \alpha\mu) M(\mu) \\ &:= U V_\alpha(\lambda) W - (\lambda - \alpha\mu) M(\mu).\end{aligned}$$

Theorem 4.3. *Let X be a Banach space. Assume that the operator L_0 defined in Eq.* (4.1) *satisfies* (H_1)–(H_5). *Then,*
(i) if for every $\lambda \in \left[\Phi_{+A} \cap \Phi_{+S(\mu)}\right]\backslash\{0\}$, there exists $T_{\lambda l}$ a left Fredholm inverse of $UV(\lambda)W$ such that $M(\mu)T_{\lambda l} \in \Lambda_X^G$, then

$$\sigma_{e_1}(L)\backslash\{0\} \subset \left[\sigma_{e_1}(A) \cup \sigma_{e_1}(S(\mu))\right]\backslash\{0\}, \text{ where } L \text{ is the closure of } L_0.$$

(ii) If for all $\alpha \in [0,1]$ and $\lambda \in \left[\Phi_{\alpha A} \cap \Phi_{\alpha S(\mu)}\right]\backslash\{0\}$, there exists $T_{\lambda l}^\alpha$ a left Fredholm inverse of $UV_\alpha(\lambda)W$ such that $M(\mu)T_{\lambda l}^\alpha \in \Lambda_X^G$, then

$$\sigma_{e_i}(L)\backslash\{0\} \subset \left[\sigma_{e_i}(\alpha A) \cup \sigma_{e_i}(\alpha S(\mu))\right]\backslash\{0\}, \text{ where } i \in \{4, 5\}.$$

$\diamondsuit$

Proof. (i) Let $\lambda \notin \sigma_{e_1}(A) \cup \sigma_{e_1}(S(\mu)) \cup \{0\}$, then $\lambda - A \in \Phi_+(X)$ and $\lambda - S(\mu) \in \Phi_+(X)$. It follows, from Theorem 3.1, that A and $S(\mu)$ are generalized weakly demicompact. Using the fact that there exists $T_{\lambda l}$ a left Fredholm inverse of $UV(\lambda)W$ such that $M(\mu)T_{\lambda l} \in \Lambda_X^G$, and when applying Corollary 4.1, we deduce that the matrix operator L is a generalized weakly demicompact. Thus, according to Theorem 3.1, $\lambda - L$ is an upper semi-Fredholm operator. So, $\lambda \notin \sigma_{e_1}(L)\backslash\{0\}$. Hence,

$$\sigma_{e_1}(L)\backslash\{0\} \subset \left[\sigma_{e_1}(A) \cup \sigma_{e_1}(S(\mu))\right]\backslash\{0\}.$$

(ii) Let $\lambda \notin \sigma_{e_4}(\alpha A) \cup \sigma_{e_4}(\alpha S(\mu)) \cup \{0\}$, then $\lambda - \alpha A \in \Phi(X)$ and $\lambda - \alpha S(\mu) \in \Phi(X)$ this follows that, $\lambda - \alpha A \in \Phi_+(X)$ and $\lambda - \alpha S(\mu) \in \Phi_+(X)$. Thus, from Theorem 3.1, we have αA and $\alpha S(\mu)$ are generalized weakly demicompact operators, for all $\alpha \in [0,1]$. Now, using the fact that there exists $T^{\alpha}_{\lambda l}$ a left Fredholm inverse of $UV_{\alpha}(\lambda)W$ such that $M(\mu)T^{\alpha}_{\lambda l} \in \Lambda^{G}_{X}$ and when applying Theorem 4.2, we obtain that the matrix operator αL is a generalized weakly demicompact for all $\alpha \in [0,1]$. Hence, by applying Theorem 3.2, $\lambda - L \in \Phi(X \times X)$. So, $\lambda \notin \sigma_{e_4}(L)\backslash\{0\}$. Thus,

$$\sigma_{e_4}(L)\backslash\{0\} \subset \left[\sigma_{e_4}(\alpha A) \cup \sigma_{e_4}(\alpha S(\mu))\right]\backslash\{0\}.$$

For the Schechter's spectrum. Let $\lambda \notin \sigma_{e_5}(\alpha A) \cup \sigma_{e_5}(\alpha S(\mu)) \cup \{0\}$, then $\lambda - \alpha A$ and $\lambda - \alpha S(\mu)$ are Fredholm with index zero. Consequently, we get $\lambda - \alpha A \in \Phi_{+}(X)$ and $\lambda - \alpha S(\mu) \in \Phi_{+}(X)$. Thus, from Theorem 3.1, we have αA and $\alpha S(\mu)$ are generalized weakly demicompact for all $\alpha \in [0,1]$. Now, using the fact that there exists $T^{\alpha}_{\lambda l}$ a left Fredholm inverse of $UV_{\alpha}(\lambda)W$ such that $M(\mu)T^{\alpha}_{\lambda l} \in \Lambda^{G}_{X}$ and when applying Theorem 4.2, we obtain that the matrix operator αL is a generalized weakly demicompact for all $\alpha \in [0,1]$. Hence, from Theorem 3.2, $\lambda - L \in \Phi(X \times X)$ with index zero. Thus, $\lambda \notin \sigma_{e_5}(L)\backslash\{0\}$.

□

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